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Constant Reference Tracking for Fast Linear Constrained Systems

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Topics

1. **Introduction.**
   which introduces main goals of the paper.
2. **Problem Description.**
   which describes the problem formulations and preliminary results.
3. **Main Results.** Including:
   (i) closed form solution to the constrained control system exploiting Explicit Model Predictive Control.
   (ii) Introducing a low complexity supervisory structure with guaranteed feasibility.
4. **Simulation Results.**
5. **Conclusion Remarks.**

1- Introduction

- Tracking problems come into view in several control applications such as motion control, robotics, chemical industry and etc. A control system in which the reference signal has to be followed by the system output is called servo-system. The servo-mechanism and tracking problem are widely studied in the literatures from different point of view.

This paper addresses the constant reference tracking problem of linear **constrained systems with fast dynamics.** Constraints satisfaction and tracking goals are guaranteed by introducing an augmented servo architecture resulting in an analytical closed-form solution. The proposed architecture has been combined with explicit model predictive control (eMPC) method to achieve optimal tracking, constraints satisfaction and piecewise closed form solution simultaneously. The proposed architecture is intuitive and gives a direct insight into the physical system and closed loop feedback properties, and therefore makes it easy to tune and select design parameters in practice. Furthermore, the piecewise closed form solution leads to a computationally efficient controller which enables it to be applicable for fast dynamic systems. The constant reference signal assumption reduces the complexity of the explicit MPC solution considerably. However, it is shown that the present method can be easily applied to applications with finite predefined set-points. Furthermore, in order to cope with non-predefined set-point applications, a convex combination based supervisor is suggested and guaranteed to be feasible under some mild assumptions.

2- Problem Formulation

Consider the discrete-time linear time invariant system:

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k) \qquad (1)$$

where $x(k) \in R^n$, $u(k) \in R^m$ and $y(k) \in R^m$ are the current state, input and output of the system respectively. Also it is assumed that the pair (A, B) is controllable and the system (1) is subject to the following constraints:

$$x(k) \in \mathbf{X} \subset R^n, \quad u(k) \in \mathbf{U} \subset R^m \qquad (2)$$

where $\mathbf{X}$ and $\mathbf{U}$ are compact polyhedra sets containing the origin in their interiors. The objective is to design a suitable controller $u(k)$ which is able to steer the system output $y(k)$ to any admissible command signal $r(k)$ while no constraint is violated.

2- Problem Formulation

- **Proposed Augmented Servo Architecture:**

The control objective of the predefined problem is threefold:

(1)- stabilizing the system,

(2)- achieving the offset-free tracking, and

(3)- handling the constraints (Feasibility).

We suggest the following control policy in which corresponding to each objective an independent portion is considered in the control signal as follows:

$$u(k) = u_{stab}(k) + u_{track}(k) + u_{constraint}(k) \quad (3)$$

2- Problem Formulation

- Regarding the control policy (3), and choosing the design parameters as followings:

$$\begin{cases} u_{stab}(k) = -K_1 x(k) \\ u_{track}(k) = K_2 S(k) \\ u_{constraint}(k) = u_c(k) \end{cases}$$

Where:

$$\begin{cases} S(k) = S(k-1) + e(k) \\ e(k) = r(k) - y(k) \end{cases}$$

- Then after some algebraic manipulations the augmented system is obtained as shown in (4):

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} A & B \\ \Gamma_1 & \Gamma_2 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0_{n\times m} \\ K_2 \end{bmatrix} r(k+1) + \begin{bmatrix} 0_{n\times 1} \\ \Delta u_c(k+1) \end{bmatrix}$$

$$y(k) = [C \;\; 0_{m\times 1}] \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \qquad \textbf{(4)}$$

Where:

$$\begin{cases} \Gamma_1 = K_1 - K_1 A - K_2 C A \\ \Gamma_2 = I_m - K_1 B - K_2 C B. \end{cases}$$

2- Problem Formulation

Now let see two key definitions:

Definition 1. *(Feasible Set $\mathbf{X_f}$)* *The feasible set $\mathbf{X_f} \subset R^n$ is defined as the set of all states $x \in R^n$ for which the constrained problem defined by (1) and (2) is feasible, i.e.*

$$\mathbf{X_f} = \{x \in R^n \,|\, \exists\, u\,(x\,(k)) \in \mathbf{U},\; x\,(k) \in \mathbf{X}\} \tag{5}$$

Definition 2. *(Maximal LQR Invariant Set $\mathbf{\Omega}_\infty$)* *For an LTI system (1) subject to the LQR control input $u(k) = -Kx(k)$, the set $\mathbf{\Omega}_\infty \subseteq R^n$ denotes the maximum invariant set of states which satisfy the constraints in (2) for all time, i.e.,*

$$\mathbf{\Omega}_\infty = \left\{ x_0 \in R^n \,\middle|\, \begin{array}{c} x\,(k) \in \mathbf{X},\;\; u\,(k) \in \mathbf{U}, \\ (A - BK)\,x\,(k) \in \mathbf{X},\; \forall k \geq 0 \end{array} \right\} \tag{6}$$

2- Problem Formulation

Remark 1. *in the case where the system is unconstrained and the pair (A, B) is controllable, then $u_c(k) = 0, \forall k = 1, 2, \ldots$ and it is always possible to find K_1 and K_2 (e.g. using LQR method) to achieve control objectives (i.e. stability and tracking). In the constrained case it might be impossible to find such a control law. This fact refers to the so-called feasibility problem (see definition 1 where $\mathbf{X_f} = \emptyset$ or $x_0 \notin \mathbf{X_f}$). This issue has been widely investigated in the literatures (see e.g. [26]).*

In the following it is assumed that the problem is feasible. i.e. $\mathbf{X_f} \neq \emptyset$

Assuming Definitions 1 and 2, and considering Remark 1, then the following theorem describes the asymptotic property of the closed loop system under constraints:

Theorem 1. *Consider the linear time invariant system (1) together with the constraint (2) and a constant command signal (r). If the control law in (3) is chosen and if $\mathbf{X_f} \neq \emptyset$ (see Def. 1), then $(y_\infty - r) \to 0$ and the followings are held:*

$$(i)\quad x_\infty = (I_n - A)^{-1} B \left[C (I_n - A)^{-1} B\right]^{-1} r,$$

$$(ii)\quad u_\infty = \left[C (I_n - A)^{-1} B\right]^{-1} r \qquad (7)$$

Proof: See paper.

2- Problem Formulation

Note that:

Theorem 1 guaranties the asymptotic property of tracking for a constant reference signal and when the corresponding system is feasible, i.e. there exist the design parameters $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ and u_c(k) for which the system (4) is stable and all constraints are satisfied. In what follows, utilizing the explicit MPC method, the parameters $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ and u_c(k) are calculated in a way that the stability and constraints satisfaction are guaranteed to be satisfied simultaneously.

3- Main Results

Let see, what is Explicit MPC ?

- In the standard MPC an open loop control problem is solved by an on-line optimization parameterized by the current state of the system. This finite-time constrained optimal control problem is formulated as:

$$J_N^*(x_0) = \min_U \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k\right) + x_N^T Q_f x_N$$ **Objective**

Constraints:

$$\begin{cases} s.t. \quad x \in \mathbf{X} \subseteq \Re^n, \ \forall k \in \{1, 2, ..., N\} \\ u \in \mathbf{U} \subseteq \Re^m, \ \forall k \in \{0, ..., N-1\} \\ x_{k+1} = A x_k + B u_k, \quad x(0) = x_0, \\ Q = Q^T \succeq 0, Q_f = Q_f^T \succeq 0, R = R^T \succ 0 \end{cases} \quad (8)$$

where, $U = \left[u_0', ..., u_{N-1}'\right]'$ is the optimization variable,

3- Main Results

- The optimization problem (8) can be easily converted to the following multi-parametric Quadratic Programming (mp-QP) form:

(mp-QP)
$$V_z^*(x) = \min_z \frac{1}{2} z^T \mathbf{H} z \qquad s.t. \;\; \mathbf{G} z \leqslant W + \mathbf{S} x \tag{9}$$

Where $z \triangleq U + \mathbf{H}^{-1}\mathbf{F}^T x$, and **H**, **G**, *W* and **S** are constant matrices of appropriate dimension.

3- Main Results

Solution to the mp-QP problem (9) and the main problem (8) is characterized by the following theorems:

Theorem 2. [18], *consider the optimization problem (9) with* $\mathbf{H} \succ 0$. *Let* $\mathbf{X} \subseteq R^n$ *be a polyhedron. Then the solution* $z^*(x)$ *and the Lagrange multiplier* $\lambda^*(x)$ *of the mp-QP problem are piecewise affine functions of the parameter* x, *and* $z^*(x)$ *is continuous.*

Theorem 3. [17], *Consider the finite time constrained problem (8). Then the set of feasible parameters* $\mathbf{X_f}$ *is convex, the optimizer* $U^* : \mathbf{X}_f \rightarrow R^s$ *is continuous and PWA function of the current state* x, *and*

$$\begin{aligned} u^*(x_0) &= F_1^r x(0) + F_2^r, \;\; if \;\; x(0) \in \mathrm{P}_r, \\ P_r &= \{x \in R^n |\, H_r x \leqslant K_r, \;\; r \in \{1, 2, ..., N_p\}\}, \\ \mathbf{X_f} &= \{\cup_{r=1}^{N_P} P_r\} \end{aligned} \qquad \textbf{(10)}$$

where $u^*(x_0) = [I_m \; 0_{m\times(s-m)}]U^*$ *denotes the first* m *components of the optimizer* U^*.

3- Main Results

- **Combining the results in this section with the proposed architecture in the previous section leads to a piecewise closed form solution of the design parameters.**

we summarize the results in the following theorem:

Theorem 4. *Consider the LTI system (1) together with the constraints (2) and constant reference signal r. If the control law (3) is chosen and the closed loop system is feasible by means of (5), then the parameters* $\boldsymbol{K_1}$, $\boldsymbol{K_2}$ *and* $\boldsymbol{u_c}(k)$ *which guarantee the tracking objective and constraints satisfaction are given as:*

$$u_c(k) = u_c(k-1) + F_2^r(k-1)$$

$$\begin{bmatrix} K_1 & K_2 \end{bmatrix} = \left[\begin{bmatrix} \mathbf{0} & I_m \end{bmatrix} - F_1^r\right] \begin{bmatrix} A - I_n & B \\ CA & CB \end{bmatrix}^{-1} \quad \mathbf{(11)}$$

where F_1^r *and* F_2^r *are given in (10).*

Proof: Please see paper

3- Main Results

- **Remark 2.** Note that the non-constant reference tracking can also be taken into account by adding the reference signal as an extra parameter in the mp-QP problem, but this may increase the complexity of the resulting controller considerably.

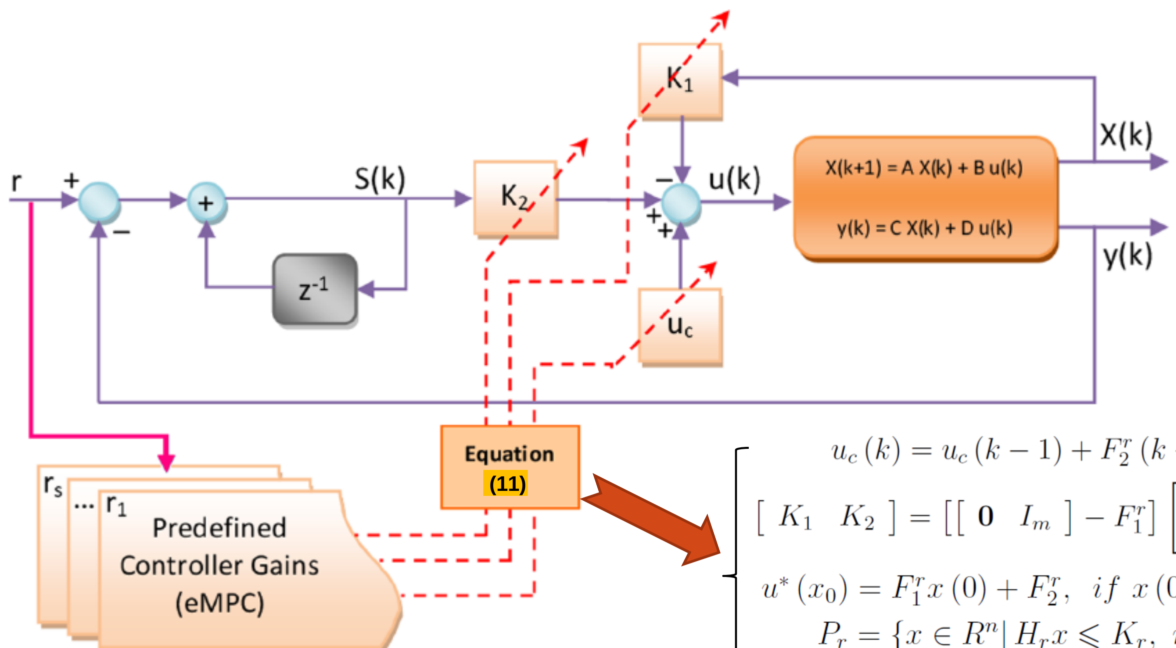
  However, as a remedy to this assumption, it is emphasized that the eMPC controller can be calculated for different reference signals independently and then by using an appropriate convex combination based supervisor the control signal is calculated for any feasible reference signal within the range of predefined reference signals. The following theorem characterizes this idea.

3- Main Results

Theorem 5. *Consider the system (1) subject to the constraints (2) and the optimization problem (8). If U_1^* and U_2^* are two feasible optimal solutions corresponding to the reference signals r_1 and r_2, then any convex combination based interpolation of U_1^* and U_2^* guarantees the constraints satisfaction (feasibility) of the optimization problem (8).*

Proof: See paper.

The following figure shows the architecture of the closed loop system.



$$
\begin{cases}
u_c(k) = u_c(k-1) + F_2^r(k-1) \\
\begin{bmatrix} K_1 & K_2 \end{bmatrix} = \left[\begin{bmatrix} \mathbf{0} & I_m \end{bmatrix} - F_1^r\right] \begin{bmatrix} A - I_n & B \\ CA & CB \end{bmatrix}^{-1} \\
u^*(x_0) = F_1^r x(0) + F_2^r, \quad if \ x(0) \in \mathrm{P}_r, \\
P_r = \{x \in R^n \mid H_r x \leqslant K_r, \ r \in \{1, 2, ..., N_p\}\}, \\
\mathbf{X_f} = \{\cup_{r=1}^{N_P} P_r\}
\end{cases}
$$

4- Simulation Results

- Consider the following discrete-time system:

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1 & -0.05 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x(k)$$ **(12)**

where the sampling time is $T_s = 0.01\ sec$ and the system (12) is subject to the following constraints:

$$\|u(k)\|_\infty \leq 1, \quad \|x(k)\|_\infty \leq 5$$ **(13)**

Constraints

4- Simulation Results

the augmented system can be obtained as

$$\xi(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.1 & -0.05 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v(k) \qquad \textbf{(14)}$$

Where: $\xi(k) = \begin{bmatrix} x(k) - x_\infty \\ u(k) - u_\infty \end{bmatrix}$

In this simulation three constant reference signals $r_1 = -5$, $r_2 = 0$ and $r_3 = 5$ are assumed and explicit controllers are calculated for each case, i.e. $Ctrl_{-5}$, $Ctrl_0$ and $Ctrl_5$. The constraints corresponding to the augmented system can be obtained for each case by using (7), (13) and (14). For example for $r = 5$ one can obtain

$$-\begin{bmatrix} 6.667 \\ 6.667 \\ 6.667 \\ 1.250 \end{bmatrix} \leqslant \xi(k) \leqslant \begin{bmatrix} 3.33 \\ 3.33 \\ 3.33 \\ 0.75 \end{bmatrix} \qquad \textbf{(15)}$$

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4- Simulation Results

- Applying the proposed approach to the augmented constrained system (14)-(15) leads to explicit solution of the controller parameters as characterized in Theorem 4. The explicit solution is defined over polyhedral regions in the state space in which the control signal is a piecewise affine function of the current state (x) and each region has own controller gains F_r^1 , F_r^2.
- In Fig.A the polyhedral regions of the augmented system for $r = 5$ and 4 = 0 are shown.

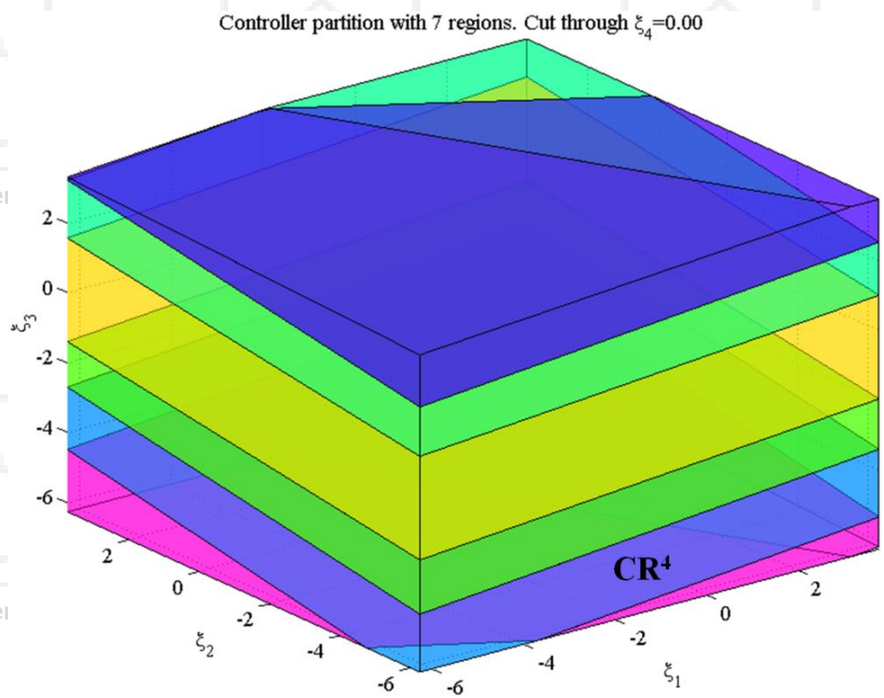


Fig.A

Polyhedral partitions of the augmented system cut through:

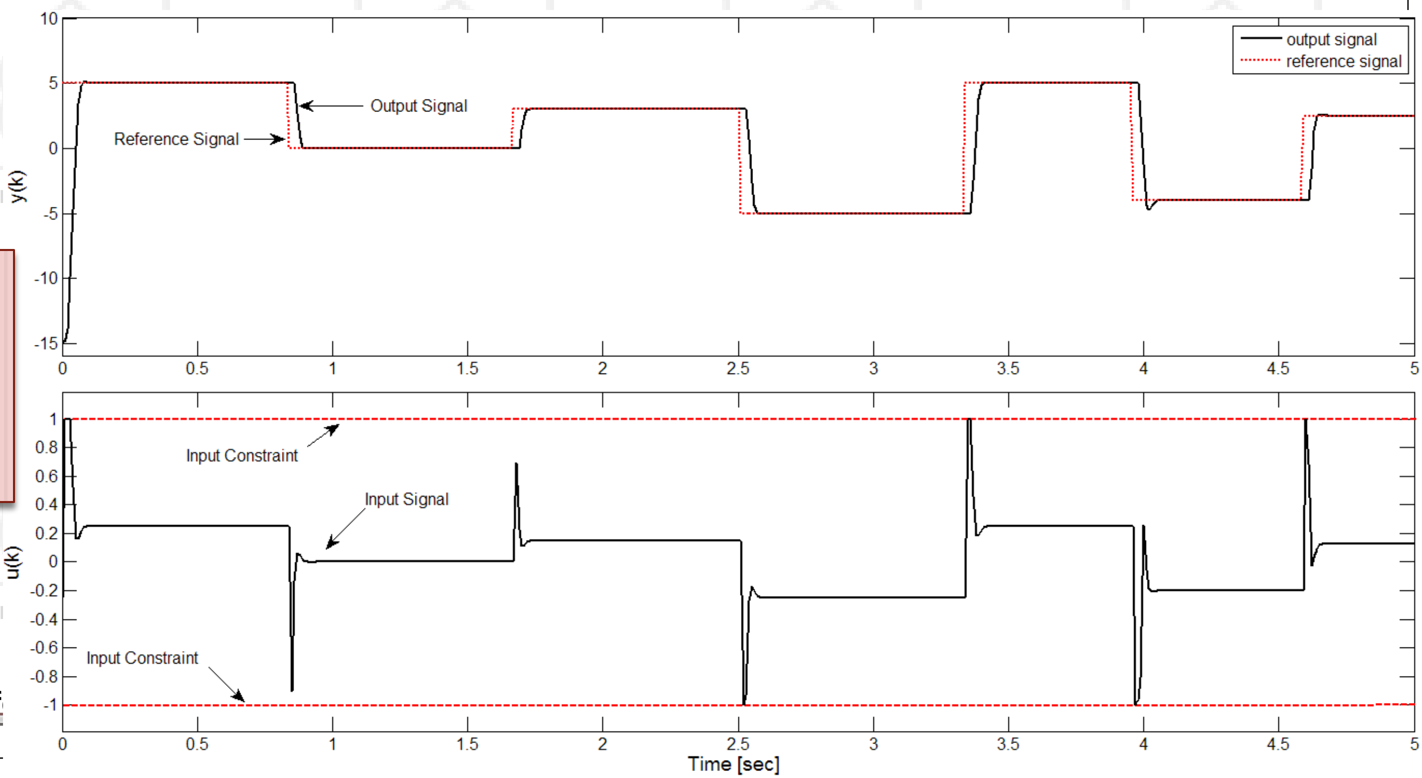
$$\xi_4 = 0 \text{ for } r = 5$$

4- Simulation Results

- Finally, corresponding to constant reference signals and according to the result in Theorem 5, a convex combination of the controllers are used to apply to the system.
- The closed loop response of the system and the input control signal are depicted in Fig.B for different command signals. According to the result it can be seen that the offset free tracking is achieved although the reference signal is different from r = ±5 and r = 0. Also it can be seen that the required input constraint $\|u(k)\|_\infty \leq 1$ is fulfilled during the simulation as it was already guaranteed in the Theorem 5.

Fig.B

Output time trajectory and associated input control signal of closed loop system



5- Conclusions:

- The constant reference tracking problem of constrained fast dynamic systems was studied.
- An augmented tracking architecture was proposed and it was shown that combining the proposed architecture with explicit model predictive control method enables us to calculate parameters of controller explicitly and hereby we would able to cope with fast dynamic system.
- Furthermore, it was shown that the proposed controller can be efficiently applied to applications with finite number of set-points by pre-computing the controller for each set-point separately.
- Finally, for non-predefined set-point applications, an interpolation based approach was suggested and proved that under some mild assumptions the interpolated controller is also feasible and guarantees the constraints satisfaction.

Thank you for attention.