In the name of God

دانشگاه زنجان

Engineering Mathematics

(Lecture # 02)

By:
Dr. Farhad Bayat
Zanjan University
Email: bayat.farhad@gmail.com

Fourier Analysis

THEOREM 1 Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the **Fourier series** of f is the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (7)$$

where the coefficients a_n and b_n in this series are defined by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad (8)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

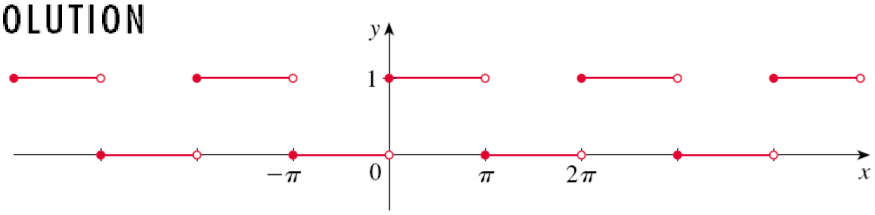
and are called the **Fourier coefficients** of f.

Fourier Analysis

EXAMPLE 1 Find the Fourier coefficients and Fourier series of the **square-wave function** f defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases}$$

SOLUTION



$f(x + 2\pi) = f(x)$ → So f is periodic with period 2π

Using the formulas for the Fourier coefficients in Definition 7, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

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and, for $n \geqslant 1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\,\frac{\sin nx}{n}\bigg]_0^{\pi} = \frac{1}{n\pi}(\sin n\pi - \sin 0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= -\frac{1}{\pi}\,\frac{\cos nx}{n}\bigg]_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - \cos 0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

|||| Note that $\cos n\pi$ equals 1 if n is even and -1 if n is odd.

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Since odd integers can be written as $n = 2k - 1$, where k is an integer, we can write the Fourier series in sigma notation as

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)x$$

How this function is equal to its Fourier series?!

Let's investigate this question graphically.

Fourier Analysis

$$S_n(x) = \underbrace{\overbrace{\frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x}^{S_3(x)}}_{S_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x} + \cdots + \frac{2}{n\pi}\sin nx$$

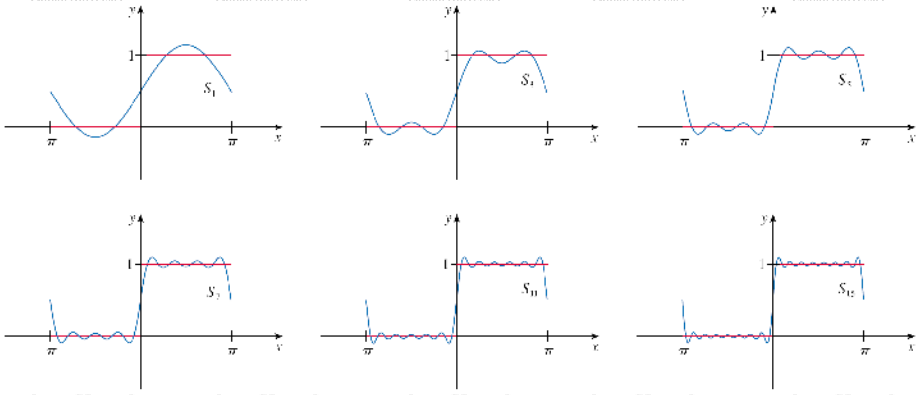


FIGURE 2 Partial sums of the Fourier series for the square-wave function

Fourier Analysis

One application of Fourier series:

Calculation of series (S=?): $S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = ?$

Let calculate f(x) at x=pi/2 :

$$f\left(\frac{\pi}{2}\right) = 1 = \left[\frac{1}{2} + \sum_{1}^{\infty} \frac{2}{(2k-1)\pi} \sin\left((2k-1)x\right)\right]_{x=\frac{\pi}{2}}$$

$$1 = \left[\frac{1}{2} + \left(\frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} - \frac{2}{7\pi} + ...\right)\right]$$

$$\frac{1}{2} = \left(\frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} - \frac{2}{7\pi} + ...\right) \Rightarrow \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = S}$$

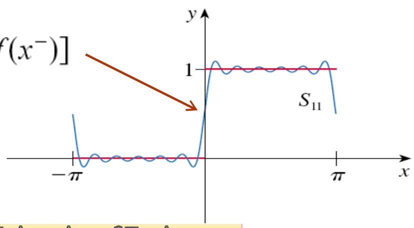
Fourier Analysis

We see that, as n increases, $S_n(x)$ becomes a better approximation to the square-wave function. It appears that the graph of $S_n(x)$ is approaching the graph of $f(x)$, except where $x = 0$ or x is an integer multiple of π. In other words, it looks as if f is equal to the sum of its Fourier series except at the points where f is discontinuous.

THEOREM 2

Fourier Convergence Theorem If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series (7) is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is

$$\tfrac{1}{2}[f(x^+) + f(x^-)]$$



Fourier Analysis

Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (8) with coefficients given by the Euler formulas (7). Theorem 2 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

Remark:

The trigonometric system {**1, cos(x), sin(x), cos(2x), sin(2x), …**} is orthogonal on the interval –pi<= x <=pi, therefore they can be used as a base set to represent any periodic function.

Exercise:

Find the Fourier series of the following function:

$$f(x) = \sqrt{2}\sin(2x) + \cos(x)$$

Questions? Discussion? Suggestions ?