In the name of God

# Engineering Mathematics

## (Lecture # 05)

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## Fourier Integral

**Main idea:**

Since, of course, many problems involve functions that are ***non-periodic and are of interest on the whole x-axis, we ask what can be done to extend the*** method of Fourier series to such functions. This idea will lead to "**Fourier integrals**."

$$\boxed{f(x) = \lim_{L \to \infty} f_L(x)}$$

## Fourier Analysis

**THEOREM**

**Fourier Integral**

*If $f(x)$ is piecewise continuous* (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

(5)
$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw.$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

(4)



## Fourier Analysis

**EXAMPLE 2**

Find the Fourier integral representation of the function

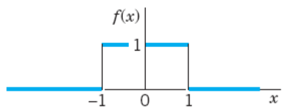


**Fig. 281.** Example 2

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

***Solution.*** From (4) we obtain

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos wv\,dv = \frac{\sin wv}{\pi w}\Bigg|_{-1}^{1} = \boxed{\frac{2\sin w}{\pi w}}$$

$$\boxed{B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wv\,dv = 0}$$

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## Fourier Analysis

and (5) gives the *answer*

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w}\,dw. \tag{6}$$

Furthermore, from (6) and Fig. 281 we obtain

$$\int_0^\infty \frac{\cos wx \sin w}{w}\,dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1, \\ \pi/4 & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases} \tag{7}$$

The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\int_0^\infty \frac{\sin w}{w}\,dw = \frac{\pi}{2}. \tag{8*}$$



## Fourier Analysis

### Approximation:

Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw \tag{9}$$

approximates the $f(x)$.

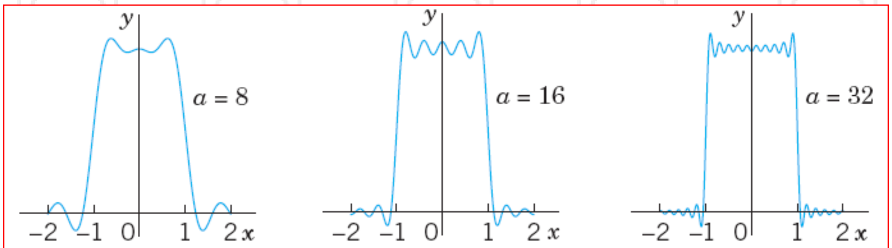


**Fig. 283.** The integral (9) for $a = 8, 16$, and 32, illustrating the development of the Gibbs phenomenon

## Fourier Analysis

**Note: Gibbs phenomenon**

Figure 283 shows oscillations near the points of discontinuity of f(x). We might expect that these oscillations disappear as **a→infinity**. *But this is not true; with increasing **a**, they are shifted closer to the points* **x=±1**. This unexpected behavior, which also occurs in connection with Fourier series, is known as the **Gibbs phenomenon.**

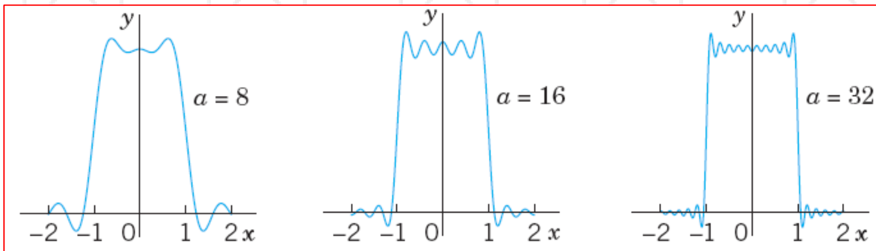


**Fig. 283.** The integral (9) for $a = 8, 16$, and 32, illustrating the development of the Gibbs phenomenon



## Fourier Analysis

### Fourier Cosine Integral and Fourier Sine Integral

**Fourier cosine integral** for even $f$

$$(10)\quad f(x) = \int_0^\infty A(w)\cos wx\, dw \quad \text{where} \quad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv.$$

**Fourier sine integral**

for *odd* $f$

$$(11)\quad f(x) = \int_0^\infty B(w)\sin wx\, dw \quad \text{where} \quad B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv.$$

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## Fourier Analysis

**EXAMPLE 3 Laplace Integrals**

$f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 284).

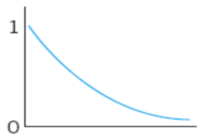


**Fig. 284.** f(x) in Example 3

***Solution.*** (a) Fourier Cosine Integral:

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv.$$

by integration by parts,

$$\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

Therefore we get:

$$A(w) = \frac{2k/\pi}{k^2+w^2}. \qquad (12)$$



## Fourier Analysis

By substituting this into the first integral in (10) we thus obtain

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw \qquad (x > 0, \quad k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw = \frac{\pi}{2k}e^{-kx} \qquad (13)$$

$(x > 0, \quad k > 0).$

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## Fourier Analysis

### (b) Fourier Sine Integral:

Similarly we have:
$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\, dv.$$

Where:
$$\int e^{-kv}\sin wv\, dv = -\frac{w}{k^2+w^2}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

Therefore:
$$\boxed{(14) \qquad B(w) = \frac{2w/\pi}{k^2+w^2}.}$$

Therefore we get:
$$\boxed{f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}\, dw.}$$



## Fourier Analysis

From this we see that

$$(15) \qquad \int_0^\infty \frac{w\sin wx}{k^2+w^2}\, dw = \frac{\pi}{2}e^{-kx}$$

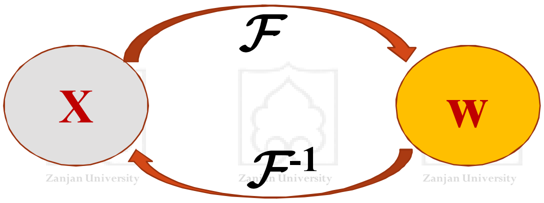
The integrals (13) and (15) are called the **Laplace integrals**.

# Fourier Cosine and Sine Transforms

An **integral transform is a transformation in the form of an integral that produces from** given functions new functions depending on a different variable.

One is mainly interested in these transforms because:

- they can be used as tools in solving ODEs, PDEs, and integral equations and
- can often be of help in handling and applying special functions.








## Fourier Cosine Transform

The Fourier cosine transform concerns ***even functions*** *f*(*x*)

$$f(x) = \int_0^\infty A(w) \cos wx \, dw, \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv.$$

we set $A(w) = \sqrt{2/\pi}\,\hat{f}_c(w)$. Then, writing $v = x$, we have

**Fourier cosine transform**

**(1a)**

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

**(1b)**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

## Fourier Analysis

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

Inverse Fourier cosine transform of $\hat{f}_c(w)$

$$\mathcal{F} \qquad \mathcal{F}^{-1}$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

Fourier cosine transform

### Fourier Sine Transform

(2a)
$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx,$$
Fourier sine transform

(2b)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw.$$
inverse Fourier sine transform



## Fourier Analysis

### Notations:

$$\mathcal{F}_c(f) = \hat{f}_c, \qquad \mathcal{F}_s(f) = \hat{f}_s$$

and $\mathcal{F}_c^{-1}$ and $\mathcal{F}_s^{-1}$ for the inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.

**EXAMPLE 1** **Fourier Cosine and Fourier Sine Transforms**

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$$

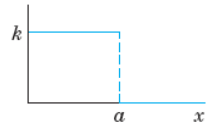



Fig. 285. $f(x)$ in Example 1

***Solution.***

From the definitions (1a) and (2a) we obtain by integration

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos aw}{w}\right).$$



Fourier Analysis

## Question?

Note that for $f(x) = k = \text{const}$ $(0 < x < \infty)$, these transforms do not exist. (Why?)

**EXAMPLE 2** **Fourier Cosine Transform of the Exponential Function**

Find $\mathcal{F}_c(e^{-x})$.

***Solution.***

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, \frac{e^{-x}}{1 + w^2}(-\cos wx + w \sin wx)\Big|_0^\infty = \frac{\sqrt{2/\pi}}{1 + w^2}.$$

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## Fourier Analysis

If $f(x)$ is **absolutely integrable** on the positive *x-axis and* ***piecewise*** **continuous** on every finite interval, then the Fourier cosine and sine transforms of *f exist.*

### Properties:

$$
(3) \quad \begin{aligned} &\text{(a)} \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \\ &\text{(b)} \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g). \end{aligned}
$$



## Fourier Analysis

**THEOREM 1**

**Cosine and Sine Transforms of Derivatives**

*Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then*

$$
(4) \quad \begin{aligned} &\textbf{(a)} \quad \mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0), \\ &\textbf{(b)} \quad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}. \end{aligned}
$$

hence by (4b)

$$
\text{(5a)} \quad \mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0).
$$

$$
\text{(5b)} \quad \mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}wf(0).
$$

## Fourier Analysis

**EXAMPLE 3**

Find the Fourier cosine transform $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.

***Solution.***

By differentiation, $(e^{-ax})'' = a^2 e^{-ax}$; thus $a^2 f(x) = f''(x)$.

From this, (5a), and the linearity (3a),

$$a^2 \mathcal{F}_c(f) = \mathcal{F}_c(f'') = -w^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -w^2 \mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}.$$

Hence $(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}$.

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right) \qquad (a > 0).$$



## Fourier Analysis

# Fourier Transform

**Final Results:**

**THEOREM 1**

**Existence of the Fourier Transform**

*If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) exists.*

(6) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx.$$

(7) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw$$

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## Fourier Analysis

**Proof:**

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv.$$

Substituting $A$ and $B$ into the integral for $f$, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw.$$



## Fourier Analysis

$$(1^*) \qquad f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right] dw.$$

Hence the integral of $F(w)$ from $w = 0$ to $\infty$ is $\frac{1}{2}$ times the integral of $F(w)$ from $-\infty$ to $\infty$.

$$(1) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv\right] dw.$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$(2) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv\right] dw = 0.$$

$\sin(wx - wv)$ is an odd function of $w$, which makes the integral in brackets an odd function of $w$.

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## Fourier Analysis

**Euler formula**

$$(3) \qquad e^{ix} = \cos x + i \sin x.$$

$$f(v) \cos(wx - wv) + if(v) \sin(wx - wv) = f(v)e^{i(wx-wv)}.$$

**complex Fourier integral**

$$(4) \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\, dv\, dw$$

Writing the exponential function in (4) as a product of exponential functions, we have

$$(5) \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)e^{-iwv}\, dv\right]}_{\hat{f}(w)} e^{iwx}\, dw.$$



## Fourier Analysis

**EXAMPLE 1** Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

***Solution.***

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{-iwx}}{-iw}\right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}}(e^{-iw} - e^{iw}).$$

As in (3) we have $e^{iw} = \cos w + i \sin w$, $e^{-iw} = \cos w - i \sin w$, and by subtraction

$$e^{iw} - e^{-iw} = 2i \sin w.$$

Substituting this in the previous formula on the right, we see that $i$ drops out and we obtain the answer

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}}\, \frac{\sin w}{w}.$$

## Fourier Analysis

**EXAMPLE 2**

Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

***Solution.***

From the definition (6) we obtain by integration

$$\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+iw)x}}{-(a+iw)} \right|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a+iw)}.$$



## Fourier Analysis

**THEOREM 2**

**Linearity of the Fourier Transform**

*The Fourier transform is a* ***linear operation****; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants $a$ and $b$, the Fourier transform of $af + bg$ exists, and*

**(8)** $$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

**THEOREM 3**

**Fourier Transform of the Derivative of $f(x)$**

*Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then*

**(9)** $$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}.$$

**(10)** $$\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}.$$

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## Fourier Analysis

**PROOF** From the definition of the Fourier transform we have

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx} \, dx.$$

Integrating by parts, we obtain

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \left[ f(x) e^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx \right].$$

Since $f(x) \to 0$ as $|x| \to \infty$, the desired result follows, namely,

$$\mathscr{F}\{f'(x)\} = 0 + iw\mathscr{F}\{f(x)\}.$$

Two successive applications of (9) give

$$\mathscr{F}(f'') = iw\mathscr{F}(f') = (iw)^2 \mathscr{F}(f).$$

Since $(iw)^2 = -w^2$, we have for the transform of the second derivative of $f$

(10) $$\mathscr{F}\{f''(x)\} = -w^2 \mathscr{F}\{f(x)\}.$$



## Fourier Analysis

**THEOREM 4**

**Convolution Theorem**

*Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then*

(12) $$\mathscr{F}(f * g) = \sqrt{2\pi}\, \mathscr{F}(f)\, \mathscr{F}(g).$$

(13) $$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx} \, dw,$$

# Questions? **Discussion?** Suggestions ?



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