In the name of God

Engineering Mathematics

(Lecture # 06)

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Fourier Analysis

SUMMARY OF CHAPTER

Fourier Analysis.

Fourier series concern **periodic functions** $f(x)$ of period $p = 2L$, that is, by definition $f(x + p) = f(x)$ for all x and some fixed $p > 0$; thus, $f(x + np) = f(x)$ for any integer n. These series are of the form

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \qquad \text{(Sec. 11.2)}$$

with coefficients, called the **Fourier coefficients** of $f(x)$, given by the Euler formulas (Sec. 11.2)

(2) $$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

SUMMARY OF CHAPTER

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$$(2) \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

SUMMARY OF CHAPTER

For period 2π we simply have (Sec. 11.1)

$$(1^*) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with the *Fourier coefficients* of $f(x)$ (Sec. 11.1)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Fourier series are fundamental in connection with periodic phenomena, particularly in models involving differential equations (Sec. 11.3, Chap, 12). If $f(x)$ is even $[f(-x) = f(x)]$ or odd $[f(-x) = -f(x)]$, they reduce to **Fourier cosine** or **Fourier sine series**, respectively (Sec. 11.2). If $f(x)$ is given for $0 \leqq x \leqq L$ only, it has two **half-range expansions** of period $2L$, namely, a cosine and a sine series (Sec. 11.2).

SUMMARY OF CHAPTER

Ideas and techniques of Fourier series extend to nonperiodic functions $f(x)$ defined on the entire real line; this leads to the **Fourier integral**

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw \tag{3} \quad \text{(Sec. 11.7)}$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv \tag{4}$$

or, in complex form (Sec. 11.9),

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw \qquad (i = \sqrt{-1}) \tag{5}$$

where

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx. \tag{6}$$

Partial Differential Equations (PDEs)

A **partial differential equation (PDE) is an equation involving one or more partial** derivatives of an (unknown) function, **call it** u. Usually one of these deals with time t *and the* remaining with space (spatial variable(s)).

The most important PDEs are:

- **The wave equations that can model the vibrating string and the vibrating membrane,**
- **The heat equation for temperature in a bar or wire, and**
- **The Laplace equation for electrostatic potentials.**

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Partial Differential Equations (PDEs)

The order of the highest derivative is called the **order of the PDE. Just as was the case for ODEs, second-order PDEs will be** the most important ones in applications.

Just as for ordinary differential equations (ODEs) we say that a PDE is **linear if it is** of the **first degree in the unknown function u and its partial derivatives**. *Otherwise we* call it **nonlinear.** **We call a *linear PDE* homogeneous if each of its terms contains either *u or one of its partial derivatives.*** Otherwise we call the equation **non-homogeneous.**

Partial Differential Equations (PDEs)

EXAMPLE 1 **Important Second-Order PDEs**

(1) (L and H)	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional wave equation*
(2) (L and H)	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional heat equation*
(3) (L and H)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	*Two-dimensional Laplace equation*
(4) (L and NH)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	*Two-dimensional Poisson equation*
(5) (L and H)	$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	*Two-dimensional wave equation*
(6) (L and H)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	*Three-dimensional Laplace equation*

(NL and H) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial z^2}\right)^2 = 0$

Partial Differential Equations (PDEs)

A **solution of a PDE in some region** ***R of the space of the independent variables is a*** function that has all the partial derivatives appearing in the PDE in some domain D containing R, *and satisfies the PDE everywhere in R.*

Often one merely requires that the function is **continuous on the boundary of** ***R***, has those **derivatives in the interior of** ***R***, *and* **satisfies the PDE in the interior of R**.

For example, the functions:

$$(7) \quad u = x^2 - y^2, \qquad u = e^x \cos y, \qquad u = \sin x \cosh y, \qquad u = \ln (x^2 + y^2)$$

which are entirely different from each other, are solutions of (3), as you may verify.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Partial Differential Equations (PDEs)

THEOREM 1

Fundamental Theorem on Superposition

If u_1 and u_2 are solutions of a ***homogeneous linear*** *PDE in some region R, then*

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that PDE in the region R.

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Partial Differential Equations (PDEs)

EXAMPLE 2 **Solving $u_{xx} - u = 0$ Like an ODE**

$u(x, y)$

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y.

Solution.

Since no y-derivatives occur, we can solve this PDE like $u'' - u = 0$.

$$u = Ae^{x} + Be^{-x}$$

Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)e^{x} + B(y)e^{-x}$$

Check the result by differentiation.

Partial Differential Equations (PDEs)

EXAMPLE 3 **Solving $u_{xy} = -u_x$ Like an ODE**

Find solutions $u = u(x, y)$ of this PDE.

Solution.

Setting $u_x = p$, → $p_y = -p$, → $p_y/p = -1$,

↓

$p = c(x)e^{-y}$ ← $\ln p = -y + \tilde{c}(x)$,

integration with respect to x,

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x)\,dx,$$

$f(x)$ and $g(y)$ are arbitrary.

Partial Differential Equations (PDEs)

Modeling: Vibrating String, Wave Equation

In this section we model a **vibrating string**, which will lead to our first important PDE, that is, equation (3) which will then be solved in the next section.

The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch, as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane.

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *Two-dimensional Laplace equation*

Partial Differential Equations (PDEs)

Problem definition:

We want to derive the PDE modeling small transverse vibrations of an elastic string, **such as a violin string**.

We place the string along the *x-axis, stretch it to length L, and fasten it* at the ends **X=0** and **X=L**.

We then distort the string, and at some instant, call it **t=0** we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection **u(x,t)** at any point *x and at any time t>=0.*

see Fig. 286.

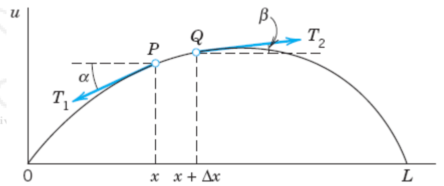

Fig. 286. Deflected string

Partial Differential Equations (PDEs)

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Partial Differential Equations (PDEs)

Derivation of the PDE of the Model ("Wave Equation") from Forces

To obtain the PDE, we consider the *forces acting on a small portion of the string (Fig. 286)*

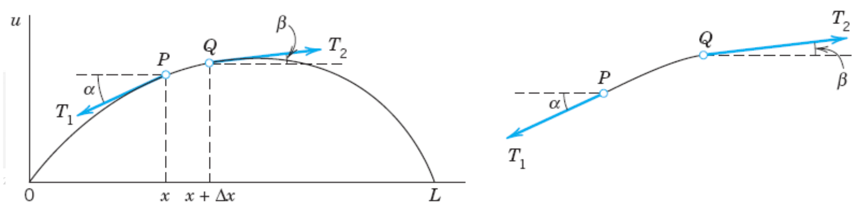

Fig. 286. Deflected string at fixed time *t*.

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point.

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Partial Differential Equations (PDEs)

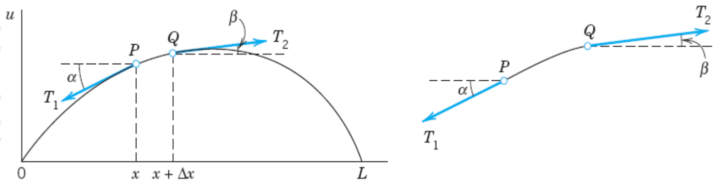

Fig. 286. Deflected string at fixed time *t*.

There is no motion in the horizontal direction.

$$(1) \qquad T_1 \cos \alpha = T_2 \cos \beta = T = \text{const.}$$

In the vertical direction By **Newton's second law**

$$T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}.$$

Partial Differential Equations (PDEs)

Using (1), we can divide this by $T_2 \cos \beta = T_1 \cos \alpha = T$, obtaining

$$(2) \qquad \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \beta - \tan \alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}.$$

We know:

$$\tan \alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x \qquad \text{and} \qquad \tan \beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}.$$

we thus have

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_x\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

Partial Differential Equations (PDEs)

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

If we let Δx approach zero, we obtain the linear PDE

(3) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho}.$$

This is called the **one-dimensional wave equation. We see that it is homogeneous and** of the second order.

Partial Differential Equations (PDEs)

Solution by Separating Variables.

Use of Fourier Series

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

Since the string is fastened at the ends $x = 0$ and $x = L$

boundary conditions

(2) (a) $u(0, t) = 0$, (b) $u(L, t) = 0$, for all $t \geqq 0$.

initial conditions

(3) (a) $u(x, 0) = f(x)$, (b) $u_t(x, 0) = g(x)$ $(0 \leqq x \leqq L)$

initial deflection *initial velocity*

Partial Differential Equations (PDEs)

We shall do this in three steps, as follows:

Step 1. By the "**method of separating variables**" or *product method*, setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3. Finally, using **Fourier series**, we compose the solutions found in Step 2 to obtain a solution of (1) satisfying both (2) and (3), that is, the solution of our model of the vibrating string.

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Partial Differential Equations (PDEs)

Step 1. Two ODEs from the Wave Equation (1)

In the **method of separating variables, or** *product method,* ***we determine solutions of the*** wave equation (1) of the form:

$$u(x, t) = F(x)G(t) \quad (4)$$

Differentiating (4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

By inserting this into the wave equation (1) we have:

$$F\ddot{G} = c^2 F''G. \Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

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Partial Differential Equations (PDEs)

$$\frac{\ddot{G}}{c^2G} = \frac{F''}{F}.$$

the left side depending only on t *and the right side only* on x.
Hence both sides must be constant.

$$\frac{\ddot{G}}{c^2G} = \frac{F''}{F} = k.$$

(5) $F'' - kF = 0$

and

(6) $\ddot{G} - c^2kG = 0.$

Partial Differential Equations (PDEs)

Step 2. Satisfying the Boundary Conditions (2)

We now determine solutions ***F*** *and* ***G*** *of (5) and (6) so that* ***u=FG*** *satisfies the boundary* conditions (2), that is,

(7) $u(0, t) = F(0)G(t) = 0, \quad u(L, t) = F(L)G(t) = 0$ for all t.

If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \neq 0$

by (7),

(8) (a) $F(0) = 0$, (b) $F(L) = 0$.

For $k = 0$ the general solution of (5) is $F = ax + b$,
and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest.

Partial Differential Equations (PDEs)

For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

from (8) we obtain $F \equiv 0$ as before (verify!).

Hence we are left with $k = -p^2$.

Then (5) becomes:

$$F'' + p^2 F = 0$$

$$F(x) = A \cos px + B \sin px.$$ general solution

Partial Differential Equations (PDEs)

$$F(x) = A \cos px + B \sin px.$$

From this and (8) we have

$$F(0) = A = 0 \qquad \text{and then} \qquad F(L) = B \sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$.

(9) $$pL = n\pi, \qquad \text{so that} \qquad p = \frac{n\pi}{L} \qquad (n \text{ integer}).$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

(10) $$F_n(x) = \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots).$$

Partial Differential Equations (PDEs)

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

$$(11^*) \qquad \ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are

$$(11) \qquad u_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots).$$

These functions are called the **eigenfunctions, or** ***characteristic functions***

$\lambda_n = cn\pi/L$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string.

The set $\{\lambda_1, \lambda_2, \cdots\}$ is called the **spectrum**.

Partial Differential Equations (PDEs)

Discussion:

Each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$

This motion is called the *nth* ***normal*** **mode of the string.**

The first normal mode is known as the *fundamental mode* (*n=1*), and the others are known as *overtones.*

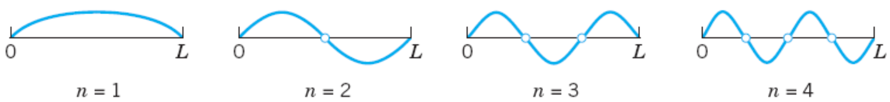

Fig. 287. Normal modes of the vibrating string

Partial Differential Equations (PDEs)

Tuning:
Tuning is done by changing the tension *T*.

$$\lambda_n/2\pi = cn/2L \qquad c = \sqrt{T/\rho}$$

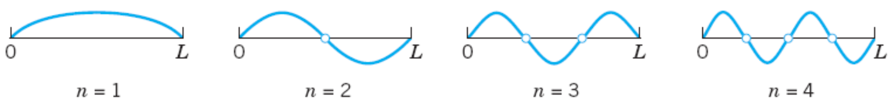

Fig. 287. Normal modes of the vibrating string

Partial Differential Equations (PDEs)

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single will generally not satisfy the initial conditions (3).

From Fundamental Theorem 1:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

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Partial Differential Equations (PDEs)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Satisfying Initial Condition (3a) (Given Initial Displacement).

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x). \qquad (0 \leqq x \leqq L). \tag{13}$$

$u(x, 0)$ becomes the **Fourier sine series** of $f(x)$.

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots. \tag{14}$$

Partial Differential Equations (PDEs)

Satisfying Initial Condition (3b) (Given Initial Velocity).

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x).$$

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

Partial Differential Equations (PDEs)

For the sake of simplicity we consider only the case when the initial velocity $g(x)$ is identically zero.

Then $\mathbf{B_n^* = 0}$, thus we get:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}. \tag{16}$$

We know:

$$\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \sin\left\{\frac{n\pi}{L}(x + ct)\right\}\right].$$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x + ct)\right\}.$$

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Partial Differential Equations (PDEs)

We can write as:

$$u(x,t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)] \tag{17}$$

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 289).

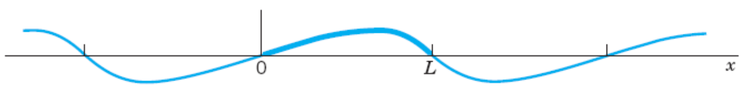

Fig. 289. Odd periodic extension of $f(x)$

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Partial Differential Equations (PDEs)

$$u(x, t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)] \qquad (17)$$

Physical Interpretation of the Solution (17).

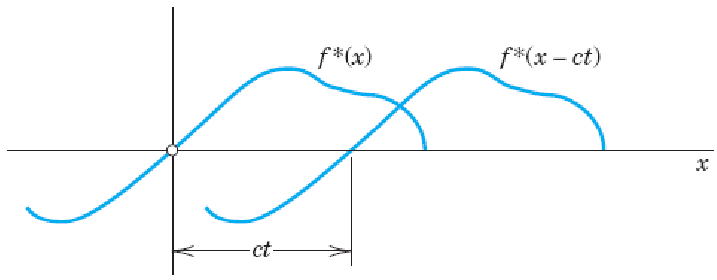

Fig. 290. Interpretation of (17)

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Questions? Discussion? Suggestions ?

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Partial Differential Equations (PDEs)

EXAMPLE 1 **Vibrating String if the Initial Deflection Is Triangular**

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 291 shows $f(x) = u(x, 0)$ at the top.)

Solution.

Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and we see that the B_n are as:

$$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right].$$

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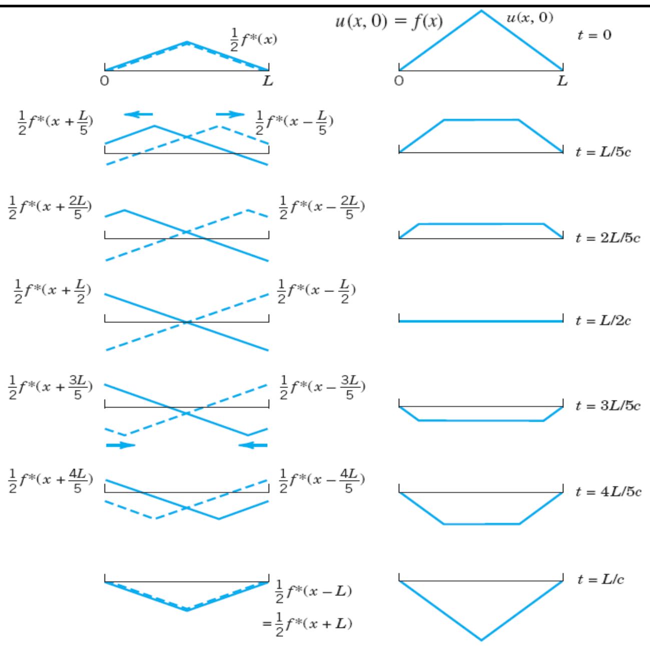

Fig. 291. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)