In the name of God

Engineering Mathematics

(Lecture # 07)

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Partial Differential Equations (PDEs)

Solution by Separating Variables.

Use of Fourier Series

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ $c^2 = \frac{T}{\rho}$

Since the string is fastened at the ends $x = 0$ and $x = L$

boundary conditions

(2) (a) $u(0, t) = 0$, (b) $u(L, t) = 0$, for all $t \geqq 0$.

initial conditions

(3) (a) $u(x, 0) = f(x)$, (b) $u_t(x, 0) = g(x)$ $(0 \leqq x \leqq L)$

initial deflection *initial velocity*

Partial Differential Equations (PDEs)

Step 1. Two ODEs from the Wave Equation (1)

In the **method of separating variables, or *product method, we determine solutions of the*** wave equation (1) of the form:

$$u(x, t) = F(x)G(t) \tag{4}$$

Differentiating (4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

By inserting this into the wave equation (1) we have:

$$F\ddot{G} = c^2 F'' G. \quad \Rightarrow \quad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

Partial Differential Equations (PDEs)

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

the left side depending only on *t and the right side only* on *x. Hence both sides must be constant.*

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

$$F'' - kF = 0 \tag{5}$$

and

$$\ddot{G} - c^2 kG = 0. \tag{6}$$

Partial Differential Equations (PDEs)

Step 2. Satisfying the Boundary Conditions (2)

We now determine solutions ***F*** *and* ***G*** *of (5) and (6) so that* ***u=FG*** *satisfies the boundary* conditions (2), that is,

(7) $$u(0, t) = F(0)G(t) = 0, \qquad u(L, t) = F(L)G(t) = 0 \qquad \text{for all } t.$$

If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \neq 0$

by (7),

(8) $$\text{(a)} \quad F(0) = 0, \qquad \text{(b)} \quad F(L) = 0.$$

For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest.

Partial Differential Equations (PDEs)

For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

from (8) we obtain $F \equiv 0$ as before (verify!).

Hence we are left with $k = -p^2$.

Then (5) becomes:

$$F'' + p^2 F = 0$$

$$\Downarrow$$

$$F(x) = A \cos px + B \sin px. \qquad \text{general solution}$$

Partial Differential Equations (PDEs)

$$F(x) = A\cos px + B\sin px.$$

From this and (8) we have

$$F(0) = A = 0 \qquad \text{and then} \qquad F(L) = B\sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$.

$$\text{(9)} \qquad pL = n\pi, \qquad \text{so that} \qquad p = \frac{n\pi}{L} \qquad (n \text{ integer}).$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$\text{(10)} \qquad F_n(x) = \sin\frac{n\pi}{L}x \qquad (n = 1, 2, \cdots).$$

Partial Differential Equations (PDEs)

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

$$\text{(11*)} \qquad \ddot{G} + \lambda_n^2 G = 0 \qquad \text{where} \qquad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n\cos\lambda_n t + B_n^*\sin\lambda_n t.$$

Hence solutions of (1) satisfying (2) are

$$\text{(11)} \qquad u_n(x,t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x \qquad (n = 1, 2, \cdots).$$

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Partial Differential Equations (PDEs)

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single will generally not satisfy the initial conditions (3).

From Fundamental Theorem 1:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Partial Differential Equations (PDEs)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Satisfying Initial Condition (3a) (Given Initial Displacement).

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x). \qquad (0 \leqq x \leqq L). \tag{13}$$

$u(x, 0)$ becomes the **Fourier sine series** of $f(x)$.

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots. \tag{14}$$

Partial Differential Equations (PDEs)

Satisfying Initial Condition (3b) (Given Initial Velocity).

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x).$$

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

(12) $$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x.$$

Partial Differential Equations (PDEs)

For the sake of simplicity we consider only the case when the initial velocity ***g(x)*** is identically zero. Then $\mathbf{B_n^* = 0}$, thus we get:

(16) $$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}.$$

We know:

$$\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \sin\left\{\frac{n\pi}{L}(x + ct)\right\}\right].$$

$$u(x, t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x - ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x + ct)\right\}.$$

From (13) ➔ $f^*(x - ct)$ $f^*(x + ct)$

Partial Differential Equations (PDEs)

We can write as:

(17) $$u(x, t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)]$$

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 289).

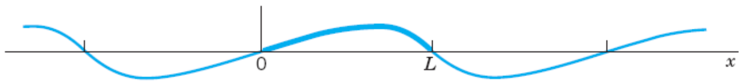

Fig. 289. Odd periodic extension of $f(x)$

Partial Differential Equations (PDEs)

(17) $$u(x, t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)]$$

Physical Interpretation of the Solution (17).

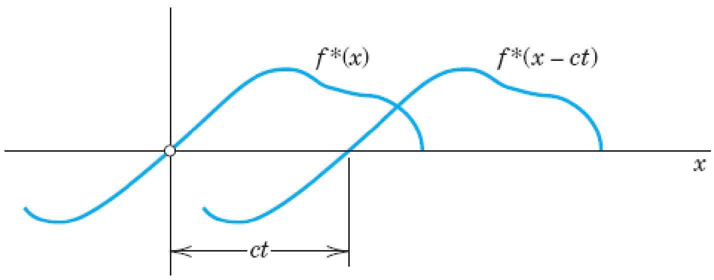

Fig. 290. Interpretation of (17)

Partial Differential Equations (PDEs)

EXAMPLE 1 Vibrating String if the Initial Deflection Is Triangular

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if} \quad 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if} \quad \frac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 291 shows $f(x) = u(x, 0)$ at the top.)

Solution.

Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and we know that the $\boldsymbol{B_n}$ are as:

$$B_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

$$u(x, t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2} \sin \frac{\pi}{L}x \cos \frac{\pi c}{L}t - \frac{1}{3^2} \sin \frac{3\pi}{L}x \cos \frac{3\pi c}{L}t + - \cdots\right].$$

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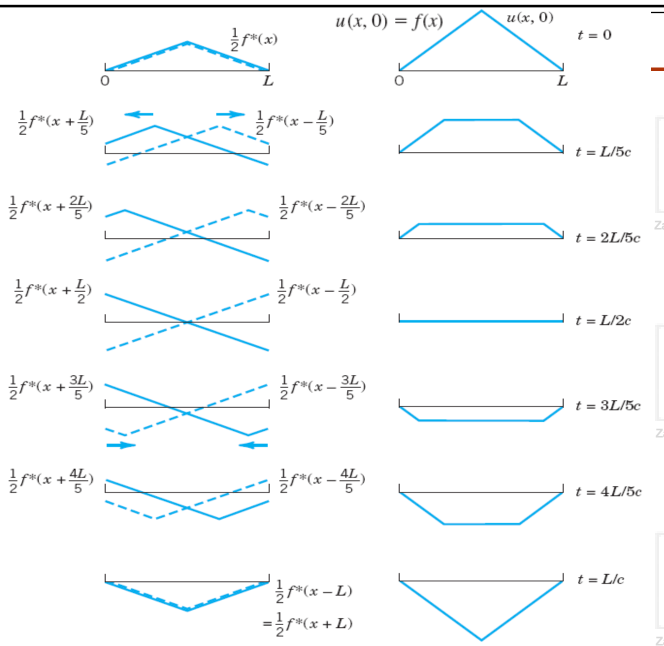

Fig. 291. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

Partial Differential Equations (PDEs)

D'Alembert's Solution of the Wave Equation:

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho},$$

by introducing the new independent variables:

$$(2) \qquad v = x + ct, \qquad w = x - ct.$$

$$v_x = 1 \text{ and } w_x = 1$$

Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

Partial Differential Equations (PDEs)

We assume that all the partial derivatives involved are continuous, so that:

$$u_{wv} = u_{vw}$$

we obtain

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}.$$

by the same procedure, we find

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww}).$$

By inserting these two results in (1) we get

$$(3) \qquad u_{vw} \equiv \frac{\partial^2 u}{\partial w \, \partial v} = 0.$$

Partial Differential Equations (PDEs)

(3) $$u_{vw} \equiv \frac{\partial^2 u}{\partial w\,\partial v} = 0.$$

The point of the present method is that (3) can be readily solved by two successive integrations, first with respect to *w and then with respect to v.*

$$\frac{\partial u}{\partial v} = h(v) \qquad \text{and} \qquad u = \int h(v)\,dv + \psi(w).$$

Here $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively.

the solution is of the form $u = \phi(v) + \psi(w)$.

by (2), we thus have

d'Alembert's solution

(4) $$u(x, t) = \phi(x + ct) + \psi(x - ct).$$

Partial Differential Equations (PDEs)

D'Alembert's Solution Satisfying the Initial Conditions

(5) (a) $u(x, 0) = f(x)$, (b) $u_t(x, 0) = g(x)$.

By differentiating (4) we have

(6) $$u_t(x, t) = c\phi'(x + ct) - c\psi'(x - ct)$$

where primes denote derivatives with respect to the *entire arguments v=x+ct and w=x-ct*, respectively.

From (4)–(6) we have

(7) $$u(x, 0) = \phi(x) + \psi(x) = f(x),$$

(8) $$u_t(x, 0) = c\phi'(x) + c\psi'(x) = g(x).$$

Partial Differential Equations (PDEs)

Dividing (8) by c and integrating with respect to x, we obtain

$$\phi(x) - \psi(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds, \qquad k(x_0) = \phi(x_0) - \psi(x_0). \tag{9}$$

If we add this to (7), then ψ drops out and division by 2 gives

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}k(x_0). \tag{10}$$

Similarly, subtraction of (9) from (7) and division by 2 gives

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2}k(x_0). \tag{11}$$

Partial Differential Equations (PDEs)

Finally we get:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{12}$$

If the initial velocity is zero, we see that this reduces to

$$u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)], \tag{13}$$

Partial Differential Equations (PDEs)

Heat Flow from a Body in Space (Heat Equation)

Physical Assumptions

1. The *specific heat* σ and the *density* ρ of the material of the body are constant. No heat is produced or disappears in the body.
2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient (cf. Sec. 9.7) of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form

(1) $$\mathbf{v} = -K \operatorname{grad} u$$

where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t.

3. The *thermal conductivity* K is constant, as is the case for homogeneous material and nonextreme temperatures.

Under these assumptions we can model heat flow as follows.

Heat Equation

(3) $$\frac{\partial u}{\partial t} = c^2 \nabla^2 u. \qquad c^2 = K/\rho\sigma$$

Partial Differential Equations (PDEs)

(3) $$\frac{\partial u}{\partial t} = c^2 \nabla^2 u. \qquad c^2 = K/\rho\sigma$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$ Laplacian of u

The heat equation is also called the **diffusion equation** because it also models chemical diffusion processes of one substance or gas into another.

Partial Differential Equations (PDEs)

Heat Equation:
Solution by Fourier Series.

As an important application of the heat equation, let us first consider the temperature in a **long thin metal bar** or **wire** of constant cross section and homogeneous material, which is oriented along the *x-axis (Fig. 294) and* ***is perfectly insulated laterally, so that heat flows* in the *x-direction only*.**

Fig. 294. Bar under consideration

Partial Differential Equations (PDEs)

Then the heat equation becomes the **one-dimensional heat equation**

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

We begin with the case in which the ends ***x = 0*** and ***x = L*** of the bar are kept at temperature zero, so that we have the **boundary conditions:**

(2) $$u(0, t) = 0, \qquad u(L, t) = 0 \qquad \text{for all } t \geqq 0.$$

Furthermore, the initial temperature in the bar at time $t = 0$ is given, say, $f(x)$, so that we have the **initial condition**

(3) $$u(x, 0) = f(x) \qquad [f(x) \text{ given}].$$

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Partial Differential Equations (PDEs)

Here we must have $f(0) = 0$ and $f(L) = 0$ because of (2).

***Step 1.* Two ODEs from the heat equation (1).**

$u(x, t) = F(x)G(t)$ → $$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}. \qquad (4)$$

The left side depends only on ***t*** *and the right side only on* ***x****, so that both sides must equal* a constant ***k***.

show that for $k = 0$ or $k > 0$ → $u \equiv 0$

Partial Differential Equations (PDEs)

For negative $k = -p^2$ we have from (4)

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2.$$

(5) $$F'' + p^2 F = 0$$

and

(6) $$\dot{G} + c^2 p^2 G = 0.$$

Partial Differential Equations (PDEs)

Step 2. **Satisfying the boundary conditions (2).**

We first solve (5). A general solution is

$$F(x) = A\cos px + B\sin px. \tag{7}$$

From the boundary conditions (2) it follows that

$$u(0, t) = F(0)G(t) = 0 \quad \text{and} \quad u(L, t) = F(L)G(t) = 0.$$

we require $F(0) = 0, F(L) = 0$ → $\begin{cases} F(0) = A = 0 \\ F(L) = B\sin pL = 0, \end{cases}$

→ $\sin pL = 0, \quad \text{hence} \quad p = \dfrac{n\pi}{L}, \quad n = 1, 2, \cdots.$

Partial Differential Equations (PDEs)

Setting $B = 1$, we thus obtain the following solutions of (5) satisfying (2):

$$F_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \cdots.$$

We now solve (6). For $p = n\pi/L$, as just obtained, (6) becomes

$$\dot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = \frac{cn\pi}{L}.$$

It has the general solution

$$G_n(t) = B_n e^{-\lambda_n^2 t}, \qquad n = 1, 2, \cdots$$

where B_n is a constant. Hence the functions

$$u_n(x, t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad (n = 1, 2, \cdots) \tag{8}$$

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Partial Differential Equations (PDEs)

Step 3. **Solution of the entire problem. Fourier series.**

To obtain a solution that also satisfies the **initial condition (3)**, we consider a series of these eigenfunctions:

$$(9) \qquad u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

From this and (3) we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x).$$

the B_n's must be the coefficients of the **Fourier sine series**,

$$(10) \qquad B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \qquad (n = 1, 2, \cdots)$$

Partial Differential Equations (PDEs)

EXAMPLE 1 Sinusoidal Initial Temperature

Find the temperature $u(x,t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100 \sin(\pi x/80)$ °C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C? First guess, then calculate. *Physical data for copper:* density 8.92 g/cm^3, specific heat 0.092 cal/(g °C), thermal conductivity 0.95 cal/(cm sec °C).

Solution.

The initial condition gives

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80} = f(x) = 100 \sin\frac{\pi x}{80}.$$

we get $B_1 = 100,\ B_2 = B_3 = \cdots = 0.$

$$c^2 = K/(\sigma\rho) = 0.95/(0.092 \cdot 8.92) = 1.158\ [\text{cm}^2/\text{sec}].$$

$$\lambda_1^2 = c^2\pi^2/L^2 \quad \Rightarrow \quad \lambda_1^2 = 1.158 \cdot 9.870/80^2 = 0.001785\ [\text{sec}^{-1}].$$

Partial Differential Equations (PDEs)

The solution (9) is

$$u(x, t) = 100 \sin \frac{\pi x}{80} e^{-0.001785t}.$$

$$100e^{-0.001785t} = 50$$

$$t = (\ln 0.5)/(-0.001785) = 388 \text{ [sec]} \approx 6.5 \text{ [min]}.$$

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Questions? Discussion? Suggestions ?

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