In the name of God

Engineering Mathematics

(Lecture # 08)

By:
Dr. Farhad Bayat
Zanjan University
Email: bayat.farhad@gmail.com

Partial Differential Equations (PDEs)

Heat PDE Equation:

(3) $$\frac{\partial u}{\partial t} = c^2 \nabla^2 u. \qquad c^2 = K/\rho\sigma$$

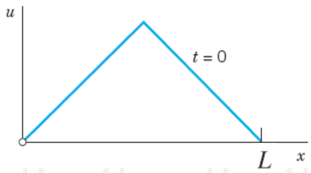
(9) $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

(10) $$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \cdots)$$

Partial Differential Equations (PDEs)

Example 3: "Triangular" Initial Temperature in a Bar

Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is



$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2, \\ L - x & \text{if } L/2 < x < L. \end{cases}$$

Solution. From (10) we get

$$B_n = \frac{2}{L}\left(\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L - x)\sin\frac{n\pi x}{L}\,dx\right). \tag{10*}$$

Integration gives $B_n = 0$ if n is even,

$$B_n = \frac{4L}{n^2\pi^2} \quad (n = 1, 5, 9, \cdots) \qquad \text{and} \qquad B_n = -\frac{4L}{n^2\pi^2} \quad (n = 3, 7, 11, \cdots).$$

Partial Differential Equations (PDEs)

Hence the solution is

$$u(x, t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left[-\left(\frac{c\pi}{L}\right)^2 t\right] - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\left[-\left(\frac{3c\pi}{L}\right)^2 t\right] + - \cdots\right].$$

Figure 295 shows that the temperature decreases with increasing t, because of the heat loss due to the cooling of the ends.

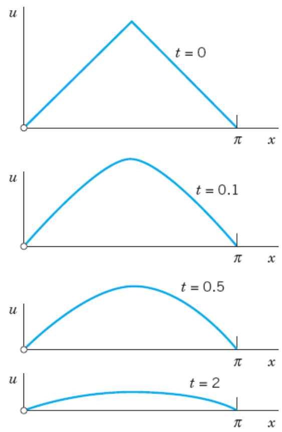


Fig. 295. Example 3. Decrease of temperature with time t for $L = \pi$ and $c = 1$

Partial Differential Equations (PDEs)

EXAMPLE 4 Bar with Insulated Ends. Eigenvalue 0

Find a solution formula of (1), (3) with (2) replaced by the condition that both ends of the bar are insulated.

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

(3) $$u(x, 0) = f(x) \qquad [f(x) \text{ given}].$$

(2) $$u(0, t) = 0, \qquad u(L, t) = 0 \qquad \text{for all } t \geqq 0.$$

Solution.

Physical experiments show that the rate of heat flow is proportional to the gradient of the temperature. Hence if the ends $x = 0$ and $x = L$ of the bar are insulated, so that no heat can flow through the ends, we have grad $u = u_x = \partial u/\partial x$ and the boundary conditions

(2*) $$u_x(0, t) = 0, \qquad u_x(L, t) = 0 \qquad \text{for all } t.$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Partial Differential Equations (PDEs)

Since $u(x, t) = F(x)G(t)$ ⇒

$$u_x(0, t) = F'(0)G(t) = 0 \text{ and } u_x(L, t) = F'(L)G(t) = 0.$$

From (7)

(7) $$F(x) = A \cos px + B \sin px.$$

$$F'(x) = -Ap \sin px + Bp \cos px,$$

$$F'(0) = Bp = 0 \qquad \text{and then} \qquad F'(L) = -Ap \sin pL = 0.$$

$$p = p_n = n\pi/L, \ (n = 0, 1, 2, \cdots)$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Partial Differential Equations (PDEs)

with $A = 1$ we get

$$F_n(x) = \cos(n\pi x/L), \ (n = 0, 1, 2, \cdots).$$

With G_n as before,

$$u_n(x,t) = F_n(x)G_n(t) = A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad (n = 0, 1, \cdots) \tag{11}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \left(\lambda_n = \frac{cn\pi}{L}\right). \tag{12}$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Partial Differential Equations (PDEs)

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \left(\lambda_n = \frac{cn\pi}{L}\right). \tag{12}$$

Its coefficients result from the initial condition (3),

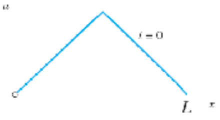
$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x),$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots. \tag{13}$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

EXAMPLE 5 **"Triangular" Initial Temperature in a Bar with Insulated Ends**

Find the temperature in the bar in Example 3, assuming that the ends are insulated (instead of being kept at temperature 0).



Solution. For the triangular initial temperature, (13) gives

$$A_0 = L/4$$

$$A_n = \frac{2}{L}\left[\int_0^{L/2} x \cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\cos\frac{n\pi x}{L}\,dx\right] = \frac{2L}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Hence the solution (12) is

$$u(x,t) = \frac{L}{4} - \frac{8L}{\pi^2}\left\{\frac{1}{2^2}\cos\frac{2\pi x}{L}\exp\left[-\left(\frac{2c\pi}{L}\right)^2 t\right] + \frac{1}{6^2}\cos\frac{6\pi x}{L}\exp\left[-\left(\frac{6c\pi}{L}\right)^2 t\right] + \cdots\right\}.$$

We see that the terms decrease with increasing t, and $u \to L/4$ as $t \to \infty$; this is the mean value of the initial temperature. This is plausible because no heat can escape from this totally insulated bar. In contrast, the cooling of the ends in Example 3 led to heat loss and $u \to 0$, the temperature at which the ends were kept.

Steady Two-Dimensional Heat Problems. Laplace's Equation

Partial Differential Equations (PDEs)

Steady Two-Dimensional Heat Problems

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

for **steady** (that is, *time-independent*) problems. Then

$$\partial u/\partial t = 0$$

the heat equation reduces to **Laplace's equation**

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(14)}$$

Partial Differential Equations (PDEs)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(14)}$$

A heat problem then consists of this PDE to be considered in some region ***R*** *of the* ***xy-plane*** *and a given boundary condition on the boundary curve C of R.*
This is a **boundary value problem (BVP).**

First BVP or **Dirichlet Problem** if u is prescribed on C ("**Dirichlet boundary condition**")

Second BVP or **Neumann Problem** if the normal derivative $u_n = \partial u/\partial n$ is prescribed on C ("**Neumann boundary condition**")

Third BVP, **Mixed BVP**, or **Robin Problem** if u is prescribed on a portion of C and u_n on the rest of C ("**Mixed boundary condition**").

Partial Differential Equations (PDEs)

Dirichlet Problem in a Rectangle R (Fig. 296).

We consider a Dirichlet problem in a rectangle R, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.

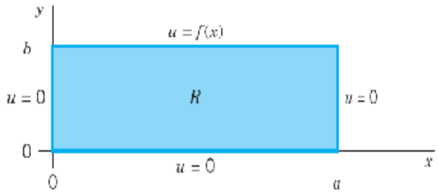


Fig. 296. Rectangle R and given boundary values

We solve this problem by separating variables.

$$u(x, y) = F(x)G(y)$$

Partial Differential Equations (PDEs)

$$u(x, y) = F(x)G(y) \rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

dividing by FG

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = -\frac{1}{G} \cdot \frac{d^2G}{dy^2} = -k.$$

From this we get

$$\frac{d^2F}{dx^2} + kF = 0,$$

Partial Differential Equations (PDEs)

the left and right boundary conditions imply

$$F(0) = 0, \quad \text{and} \quad F(a) = 0.$$

This gives $k = (n\pi/a)^2$ and corresponding nonzero solutions

$$F(x) = F_n(x) = \sin \frac{n\pi}{a} x, \quad n = 1, 2, \cdots. \tag{15}$$

The ODE for G with $k = (n\pi/a)^2$ then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0.$$

Solutions are

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

Partial Differential Equations (PDEs)

the boundary condition $u = 0$ on the lower side of R implies that

$$G_n(0) = 0;$$

$$G_n(0) = A_n + B_n = 0 \text{ or } B_n = -A_n.$$

$$G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2A_n \sinh \frac{n\pi y}{a}.$$

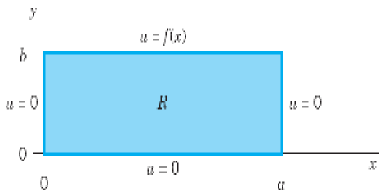


Fig. 296. Rectangle R and given boundary values

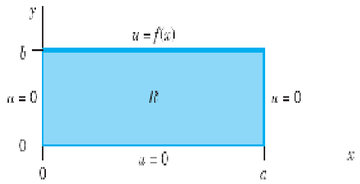
From this and (15), writing $2A_n = A_n^*$, we obtain as the **eigenfunctions**

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}. \tag{16}$$

Partial Differential Equations (PDEs)

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}. \quad (16)$$

These solutions satisfy the boundary condition $u = 0$ on the left, right, and lower sides.



To get a solution also satisfying the boundary condition $u(x, b) = f(x)$ on the upper side, we consider the infinite series

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y).$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Partial Differential Equations (PDEs)

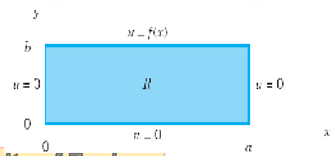
with $y = b$ we obtain

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}.$$

We can write this in the form

$$u(x, b) = \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a} \right) \sin \frac{n\pi x}{a}.$$

$u(x, b) = f(x)$ → $$b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx.$$



Lecturer: Dr Farhad Bayat, University of Zanjan.

Partial Differential Equations (PDEs)

We get:

(17) $$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

(18) $$A_n^* = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx.$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Questions? **Discussion?** Suggestions ?

Lecturer: Dr Farhad Bayat, University of Zanjan.