In the name of God

دانشگاه زنجان

Engineering Mathematics

(Lecture # 09)

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Partial Differential Equations (PDEs)

Steady Two-Dimensional Heat Problems. Laplace's Equation

Partial Differential Equations (PDEs)

Steady Two-Dimensional Heat Problems

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

for **steady** (that is, *time-independent*) problems. Then

$$\partial u/\partial t = 0$$

the heat equation reduces to **Laplace's equation**

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (14)$$

Partial Differential Equations (PDEs)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (14)$$

A heat problem then consists of this PDE to be considered in some region ***R*** *of the* ***xy-plane*** *and a given boundary condition on the boundary curve C of R*.
This is a **boundary value problem (BVP).**

First BVP or **Dirichlet Problem** if u is prescribed on C ("**Dirichlet boundary condition**")

Second BVP or **Neumann Problem** if the normal derivative $u_n = \partial u/\partial n$ is prescribed on C ("**Neumann boundary condition**")

Third BVP, **Mixed BVP**, or **Robin Problem** if u is prescribed on a portion of C and u_n on the rest of C ("**Mixed boundary condition**").

Partial Differential Equations (PDEs)

Dirichlet Problem in a Rectangle R (Fig. 296).

We consider a Dirichlet problem in a rectangle R, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.

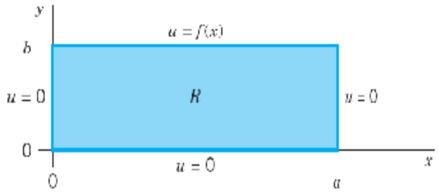


Fig. 296. Rectangle R and given boundary values

We solve this problem by separating variables.

$$u(x, y) = F(x)G(y)$$

Partial Differential Equations (PDEs)

$$u(x, y) = F(x)G(y) \Rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

dividing by FG

$$\frac{1}{F} \cdot \frac{d^2F}{dx^2} = -\frac{1}{G} \cdot \frac{d^2G}{dy^2} = -k.$$

From this we get

$$\frac{d^2F}{dx^2} + kF = 0, \quad \xrightarrow{k = p^2} \quad F(x) = A \cos px + B \sin px.$$

Partial Differential Equations (PDEs)

the left and right boundary conditions imply

$$F(0) = 0, \quad \text{and} \quad F(a) = 0.$$

This gives $k = (n\pi/a)^2$ and corresponding nonzero solutions

(15) $$F(x) = F_n(x) = \sin \frac{n\pi}{a} x, \quad n = 1, 2, \cdots.$$

The ODE for G with $k = (n\pi/a)^2$ then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0.$$

Solutions are

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

Partial Differential Equations (PDEs)

the boundary condition $u = 0$ on the lower side of R implies that

$$G_n(0) = 0;$$

$$G_n(0) = A_n + B_n = 0 \text{ or } B_n = -A_n.$$

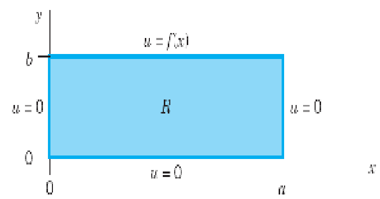


Fig. 296. Rectangle R and given boundary values

$$G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2A_n \sinh \frac{n\pi y}{a}.$$

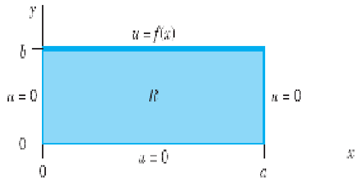
From this and (15), writing $2A_n = A_n^*$, we obtain as the **eigenfunctions**

(16) $$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

Partial Differential Equations (PDEs)

(16) $$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

These solutions satisfy the boundary condition $u = 0$ on the left, right, and lower sides.



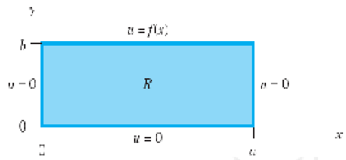
To get a solution also satisfying the boundary condition $u(x, b) = f(x)$ on the upper side, we consider the infinite series

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y).$$

Partial Differential Equations (PDEs)

with $y = b$ we obtain

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}.$$



We can write this in the form

$$u(x, b) = \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a}.$$

$u(x, b) = f(x)$ ➡ $$b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

Partial Differential Equations (PDEs)

We get:

(17) $$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

(18) $$A_n^* = \frac{2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

Partial Differential Equations (PDEs)

Heat Equation: Modeling Very Long Bars

Heat Equation

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The results in the last section can be extended to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 100 m). Then the role of Fourier series in the solution process will be taken by **Fourier integrals.**

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Partial Differential Equations (PDEs)

Let assume a bar that extends to infinity on both sides (and is laterally insulated as before).
Then **we do not have boundary conditions**, but only the **initial condition:**

(2) $$u(x, 0) = f(x) \quad (-\infty < x < \infty)$$

where $f(x)$ is the given initial temperature of the bar.

To solve this problem, we start as in the last section, substituting

$$u(x, t) = F(x)G(t)$$

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Partial Differential Equations (PDEs)

***Step 1.* Two ODEs from the heat equation (1).**

$$u(x, t) = F(x)G(t) \Rightarrow \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2.$$

For negative $k = -p^2$

(3) $$F'' + p^2 F = 0$$

and

(4) $$\dot{G} + c^2 p^2 G = 0$$

Partial Differential Equations (PDEs)

(3) $$F'' + p^2F = 0$$

and

(4) $$\dot{G} + c^2p^2G = 0$$

Solutions are

$$F(x) = A\cos px + B\sin px \qquad G(t) = e^{-c^2p^2t},$$

Hence a solution of (1) is

(5) $$u(x, t; p) = FG = (A\cos px + B\sin px)\, e^{-c^2p^2t}.$$

Partial Differential Equations (PDEs)

Remark:

Here we had to choose the separation constant k negative, $k = -p^2$, because positive values of k would lead to an increasing exponential function in (5), which has no physical meaning.

Goal:

We need to find A and B to complete the answer.
In the following we will use **Fourier Integral**.

(5) $$u(x, t; p) = FG = (A\cos px + B\sin px)\, e^{-c^2p^2t}.$$

Partial Differential Equations (PDEs)

Use of Fourier Integrals

Since *f (x)* in (2) is not assumed to be periodic and is defined from -∞ to +∞, it is natural to use **Fourier integrals** instead of **Fourier series**.

Also, since **A** *and* **B** *in (5)* **are arbitrary** *and we may regard them as functions of p, writing*

$$A = A(p) \text{ and } B = B(P).$$

Then we get a series of solutions, as:

$$u(x, t; p) = [A(p) \cos px + B(p) \sin px]\, e^{-c^2p^2t}$$

Partial Differential Equations (PDEs)

Note that, p can be changed continuously, therefore we choose *u(x,t)* as:

$$(6) \qquad u(x, t) = \int_0^\infty u(x, t; p)\, dp = \int_0^\infty [A(p) \cos px + B(p) \sin px]\, e^{-c^2p^2t}\, dp$$

Provided that, this integral exists and can be differentiated twice with respect to *x and once with respect to t.*

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Partial Differential Equations (PDEs)

Determination of $A(p)$ and $B(p)$ from the Initial Condition.

From (6) and (2) we get

$$(7) \qquad u(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\,dp = f(x).$$

Comparing (7) with **Fourier Integral**, we can obtain:

$$(8) \qquad A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv, \qquad B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv.$$

Partial Differential Equations (PDEs)

$$(8) \qquad A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv, \qquad B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv.$$

$$(6) \qquad u(x,t) = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]\,e^{-c^2p^2t}\,dp$$

Substituting A(p) and B(p) into the (6), we get:

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\,e^{-c^2p^2t}\,dv\right] dp.$$

we may reverse the order of integration, we obtain

$$(9) \qquad u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^\infty e^{-c^2p^2t}\cos(px - pv)\,dp\right] dv.$$

Partial Differential Equations (PDEs)

(9) $$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2p^2t}\cos(px - pv)\,dp\right]dv.$$

Then we can evaluate the inner integral by using the formula

(10) $$\int_0^{\infty} e^{-s^2}\cos 2bs\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}.$$

Derivation of Eq. (10) will be given in later sections.

If we choose:

$c^2p^2t=s^2$,
$px-pv = 2bs$,

$\Rightarrow$ $p = s/(c\sqrt{t})$, $b = \dfrac{x-v}{2c\sqrt{t}}$.

Partial Differential Equations (PDEs)

so that (10) becomes

$$\int_0^{\infty} e^{-c^2p^2t}\cos(px - pv)\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}.$$

By inserting this result into (9) we obtain the representation

(11) $$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}dv.$$

Taking $z = (v-x)/(2c\sqrt{t})$ as a variable of integration, we get the alternative form

(12) $$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})\,e^{-z^2}\,dz.$$

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Partial Differential Equations (PDEs)

EXAMPLE 1 **Temperature in an Infinite Bar**

Find the temperature in the infinite bar if the initial temperature is (Fig. 298)

$$f(x) = \begin{cases} U_0 = \text{const} & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| > 1. \end{cases}$$

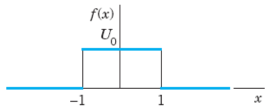

Fig. 298. Initial temperature in Example 1

Solution.

From (11) we have

$$u(x, t) = \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv.$$

Taking $z = (v - x)/(2c\sqrt{t})$ as a variable of integration, we get

$$u(x, t) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/(2c\sqrt{t})}^{(1-x)/(2c\sqrt{t})} e^{-z^2} dz$$

Partial Differential Equations (PDEs)

$$u(x, t) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/(2c\sqrt{t})}^{(1-x)/(2c\sqrt{t})} e^{-z^2} dz$$

We mention that this integral **is not an elementary function**, but **can be expressed in terms of the error function**, whose values have been tabulated.

Figure 299 shows u(x,t) for $U_0 = 100°\text{C}, c^2 = 1\ \text{cm}^2/\text{sec}$, and several values of t.

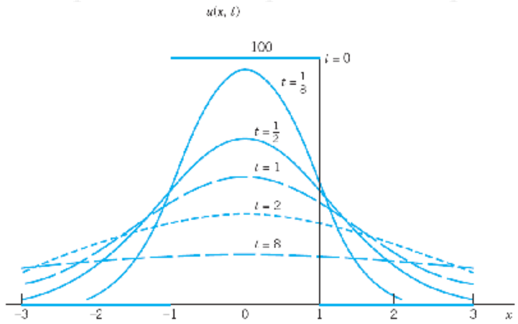

Fig. 299. Solution $u(x, t)$ in Example 1 for $U_0 = 100°\text{C}$, $c^2 = 1\ \text{cm}^2/\text{sec}$, and several values of t

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Questions? **Discussion?** Suggestions ?

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