In the name of God

Engineering Mathematics

(Lecture # 10)

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Partial Differential Equations (PDEs)

Heat Equation: Very Long Bars

Heat Equation

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The results in the last section can be extended to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 100 m). Then the role of Fourier series in the solution process will be taken by **Fourier integrals.**

Partial Differential Equations (PDEs)

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(2) $$u(x, 0) = f(x) \quad (-\infty < x < \infty)$$

$$u(x, t) = F(x)G(t)$$

$$u(x, t; p) = [A(p) \cos px + B(p) \sin px]\, e^{-c^2p^2t}$$

Using Fourier Integral.
And Initial condition $u(x, 0) = f(x)$

(9) $$u(x, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2p^2t} \cos(px - pv)\, dp\right] dv.$$

(11) $$u(x, t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v) \exp\left\{-\frac{(x - v)^2}{4c^2t}\right\} dv.$$

Remember these Results (9)&(10)!

Partial Differential Equations (PDEs)

Use of Fourier Transforms

The **Fourier transform** is closely related to the **Fourier integral.**

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\, dw$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\, dx.$$

we can use Fourier transforms for solving our present or similar problems. The **Fourier transform** applies to problems **concerning the entire axis**, and the **Fourier cosine** and **sine transforms** to problems involving the **positive half-axis**.

Partial Differential Equations (PDEs)

EXAMPLE 2 **Temperature in the Infinite Bar in Example 1**

Solve Example 1 using the Fourier transform.

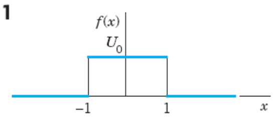

Fig. 298. Initial temperature in Example 1

Solution.

The problem consists of the heat equation (1) and the initial condition (2),

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(2) $$u(x, 0) = f(x) \quad (-\infty < x < \infty)$$

which in this example is

$$f(x) = U_0 = \text{const} \quad \text{if } |x| < 1 \qquad \text{and 0 otherwise.}$$

Our strategy is to take the Fourier transform **with respect to *x*** *and then to solve the resulting* ***ODE in t.***

Partial Differential Equations (PDEs)

Let $\hat{u} = \mathcal{F}(u)$ denote the Fourier transform of u, ***regarded as a function of x.***

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \;\Rightarrow\; \mathcal{F}(u_t) = c^2\mathcal{F}(u_{xx}) = c^2(-w^2)\mathcal{F}(u) = -c^2w^2\hat{u}.$$

On the left, assuming that we may interchange the order of differentiation and integration, we have

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-iwx}\,dx = \frac{\partial \hat{u}}{\partial t}.$$

Therefore:

$$\frac{\partial \hat{u}}{\partial t} = -c^2w^2\hat{u}.$$

$u(x, t)$

$\hat{u}(w, t)$

Since this equation involves only a derivative with respect to *t but none with respect to w, this is a first-order* ***ODE, with t as the independent variable and*** w ***as a parameter.***

Partial Differential Equations (PDEs)

We get the general solution

$$\frac{\partial \hat{u}}{\partial t} = -c^2 w^2 \hat{u}. \quad \Rightarrow \quad \hat{u}(w, t) = C(w) e^{-c^2 w^2 t}$$

Initial Condition: $u(x, 0) = f(x)$

$$\hat{u}(w, 0) = C(w) = \hat{f}(w) = \mathscr{F}(f). \quad \Rightarrow \quad \hat{u}(w, t) = \hat{f}(w) e^{-c^2 w^2 t}.$$

Now inverse Fourier gives the solution:

$$(14) \qquad u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{-c^2 w^2 t} e^{iwx}\, dw.$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iwv}\, dv.$$

Partial Differential Equations (PDEs)

Combining and assuming that we may invert the order of integration, we then obtain:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} e^{-c^2 w^2 t} e^{i(wx - wv)} dw \right] dv.$$

By the Euler formula

$$e^{-c^2 w^2 t} e^{i(wx - wv)} = e^{-c^2 w^2 t} \cos(wx - wv) + i e^{-c^2 w^2 t} \sin(wx - wv).$$

We see that its imaginary part is an odd function of w, so that its integral is 0.

The real part is an even function of w, so that

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{0}^{\infty} e^{-c^2 w^2 t} \cos(wx - wv)\, dw \right] dv.$$

This agrees with (9) (with $p = w$) and leads to the further formulas (11) and (13).

Partial Differential Equations (PDEs)

EXAMPLE 3 Solution in Example 1 by the Method of Convolution

Solve the heat problem in Example 1 by the method of convolution.

Solution. The beginning is as in Example 2 and leads to (14), that is,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2 w^2 t} e^{iwx}\, dw. \tag{15}$$

Since, by the definition of convolution

$$\mathscr{F}(f * g) = \sqrt{2\pi}\, \mathscr{F}(f)\, \mathscr{F}(g). \quad \Rightarrow \quad (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx}\, dw,$$

We recognize that this is in the form of **convolution** as:

$$u(x,t) = (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx}\, dw \tag{16}$$

where

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2 w^2 t}. \tag{17}$$

Partial Differential Equations (PDEs)

by the definition of convolution

$$u(x,t) = (f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x-p)\, dp, \tag{18}$$

as our next and last step we must determine the inverse Fourier transform g of $\hat{g}$.

$g(x)$ **?**

Where:

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2 w^2 t}.$$

Partial Differential Equations (PDEs)

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2w^2t}.$$ $g(x)$?

We know (using Fourier Table):

$$\mathscr{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$$

With $c^2t = 1/(4a)$ or $a = 1/(4c^2t)$,

$$\mathscr{F}(e^{-x^2/(4c^2t)}) = \sqrt{2c^2t}\, e^{-c^2w^2t} \Rightarrow \mathscr{F}(e^{-x^2/(4c^2t)}) = \sqrt{2c^2t}\,\sqrt{2\pi}\,\hat{g}(w).$$

Hence $\hat{g}$ has the inverse

$$g(t) = \frac{1}{\sqrt{2c^2t}\,\sqrt{2\pi}} e^{-x^2/(4c^2t)}.$$

Partial Differential Equations (PDEs)

Using (18) and replacing *x* by *x-p* in *g(t)* we get:

$$u(x, t) = (f * g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left\{-\frac{(x-p)^2}{4c^2t}\right\} dp. \quad (19)$$

This solution formula of our problem agrees with (11).

Partial Differential Equations (PDEs)

EXAMPLE 4 Fourier Sine Transform Applied to the Heat Equation

If a laterally insulated bar extends from $x = 0$ to infinity, we can use the Fourier sine transform. We let the initial temperature be $u(x, 0) = f(x)$ and impose the boundary condition $u(0, t) =$

Solution.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = 0.$$
$$u(x, 0) = f(x)$$

since $f(0) = u(0, 0) = 0,$

(Fourier w.r.t ➔ x)

$$\mathcal{F}_s(u_t) = \frac{\partial \hat{u}_s}{\partial t} = c^2 \mathcal{F}_s(u_{xx}) = -c^2 w^2 \mathcal{F}_s(u) = -c^2 w^2 \hat{u}_s(w, t).$$

Using:

$$\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0),$$

$$\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}.$$

Partial Differential Equations (PDEs)

$$\frac{\partial \hat{u}_s}{\partial t} = -c^2 w^2 \hat{u}_s(w, t).$$

This is a first-order ODE

$$\hat{u}_s(w, t) = C(w) e^{-c^2 w^2 t}.$$

From the initial condition $u(x, 0) = f(x)$ we have $\hat{u}_s(w, 0) = \hat{f}_s(w) = C(w)$. Hence

$$\hat{u}_s(w, t) = \hat{f}_s(w) e^{-c^2 w^2 t}.$$

Taking the inverse Fourier sine transform and substituting

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \sin wp \, dp$$

Partial Differential Equations (PDEs)

we then obtain the solution formula

$$u(x, t) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(p)\sin wp\, e^{-c^2w^2t}\sin wx\, dp\, dw. \tag{20}$$

Figure 300 shows (20) with $c = 1$ for $f(x) = 1$ if $0 \leqq x \leqq 1$ and 0 otherwise, graphed over the xt-plane for $0 \leqq x \leqq 2$, $0.01 \leqq t \leqq 1.5$. Note that the curves of $u(x, t)$ for constant t resemble those in Fig. 299.

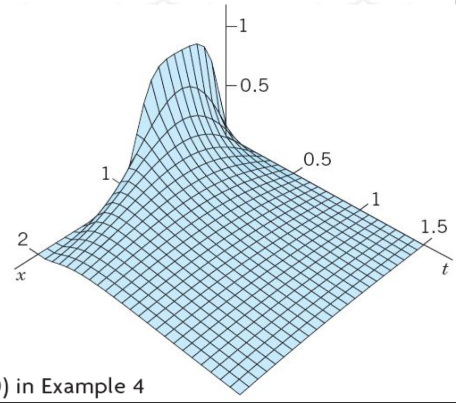

Fig. 300. Solution (20) in Example 4

Questions? Discussion? Suggestions ?

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