In the name of God

دانشگاه زنجان

Engineering Mathematics

(Lecture # 11)

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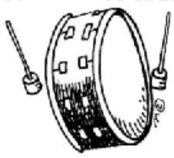
Partial Differential Equations (PDEs)

Modeling: Membrane, Two-Dimensional Wave Equation



Physical Assumptions

1. The mass of the membrane per unit area is constant (“**homogeneous membrane**”).
2. The membrane is stretched and then fixed along its entire boundary in the *xy-plane.*
3. The deflection u(x,y,t) of the membrane during the motion is small **compared to the size of the membrane**, and all angles of inclination are small.

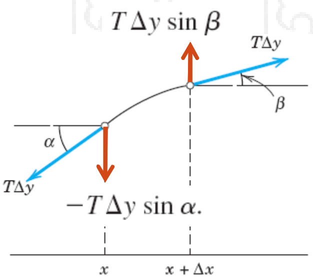
Partial Differential Equations (PDEs)

Remark:

The tension ***T*** *is the force per unit length.*

These components along the right side and the left side are (Fig. 301), respectively,

$$T\,\Delta y \sin\beta \qquad \text{and} \qquad -T\,\Delta y \sin\alpha.$$



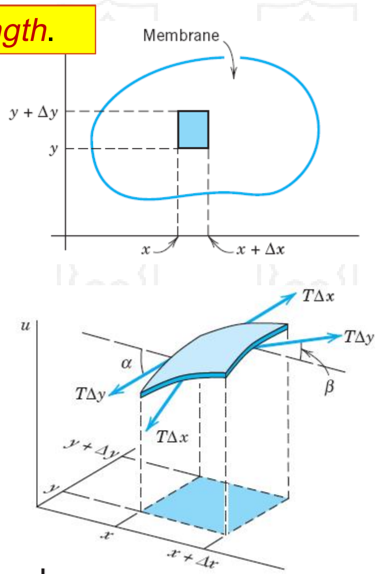


Fig. 301. Vibrating membrane

Partial Differential Equations (PDEs)

Remark:

Since the **angles are small**, then:

$$\sin\alpha \approx \tan\alpha$$
$$\sin\beta \approx \tan\beta$$

$$(1)\qquad T\,\Delta y(\sin\beta - \sin\alpha) \approx T\,\Delta y(\tan\beta - \tan\alpha) = T\,\Delta y\,[u_x(x+\Delta x, y_1) - u_x(x, y_2)]$$

where: y_1 and y_2 are values between y and $y + \Delta y$.

Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is:

$$(2)\qquad T\,\Delta x\,[u_y(x_1, y+\Delta y) - u_y(x_2, y)]$$

where: x_1 and x_2 are values between x and $x + \Delta x$.

Partial Differential Equations (PDEs)

Newton's Second Law Gives the PDE of the Model.

M * a = F

$$\rho \Delta x\, \Delta y \frac{\partial^2 u}{\partial t^2} = T\Delta y[u_x(x+\Delta x, y_1) - u_x(x, y_2)] + T\Delta x[u_y(x_1, y+\Delta y) - u_y(x_2, y)]$$

here ρ is the mass of the undeflected membrane per unit area, and $\Delta A = \Delta x\, \Delta y$

Partial Differential Equations (PDEs)

Division by $\rho \Delta x\, \Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x+\Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y+\Delta y) - u_y(x_2, y)}{\Delta y}\right]$$

If we let Δx and Δy approach zero, we obtain the PDE of the model

(3) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad c^2 = \frac{T}{\rho}.$$

OR

(3′) $$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

This PDE is called the **two-dimensional wave equation**.

Partial Differential Equations (PDEs)

Rectangular Membrane. Double Fourier Series

The model of the vibrating membrane for obtaining the displacement $u(x, y, t)$ of a point (x, y) of the membrane from rest $(u = 0)$ at time t is

(1)	$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
(2)	$u = 0$ on the boundary
(3a)	$u(x, y, 0) = f(x, y)$
(3b)	$u_t(x, y, 0) = g(x, y)$.

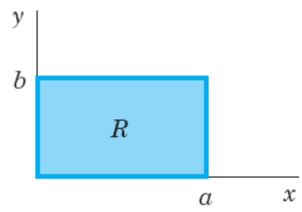


Fig. 302. Rectangular membrane

Partial Differential Equations (PDEs)

Solution Steps:

Step 1.

$u(x, y, t) = F(x, y)G(t)$ → $F(x, y) = H(x)Q(y)$

↓

Two ODEs for ***H(x)*** *and* ***Q(y).***

Step 2.

Determine **eigenfunctions u$_{mn}$**s, that satisfy the boundary condition (2).

Step 3.

Determine **whole solution u(x,y,t)** that satisfies the initial conditions (3a) and (3b).

Partial Differential Equations (PDEs)

Step 1. Three ODEs From the Wave Equation (1)

$u(x, y, t) = F(x, y)G(t)$ → $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$$

divide both sides by c^2FG:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}).$$

Partial Differential Equations (PDEs)

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}).$$

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2.$$

"time function" $G(t)$

(4) $\ddot{G} + \lambda^2 G = 0$ where $\lambda = c\nu$,

"amplitude function" $F(x, y)$

(5) $F_{xx} + F_{yy} + \nu^2 F = 0.$

The PDE (5) is called the *two-dimensional* ***Helmholtz*** **equation.**

Partial Differential Equations (PDEs)

$$F(x, y) = H(x)Q(y) \rightarrow F_{xx} + F_{yy} + \nu^2 F = 0.$$

$$\frac{d^2H}{dx^2} Q = -\left(H \frac{d^2Q}{dy^2} + \nu^2 HQ\right)$$

divide both sides by HQ

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right)$$

Partial Differential Equations (PDEs)

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right)$$

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2.$$

(6) $$\frac{d^2H}{dx^2} + k^2 H = 0$$

and

(7) $$\frac{d^2Q}{dy^2} + p^2 Q = 0$$

$p^2 = \nu^2 - k^2.$

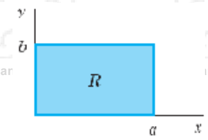
Partial Differential Equations (PDEs)

Step 2. Satisfying the Boundary Condition

General solutions of (6) and (7) are

$$\frac{d^2H}{dx^2} + k^2H = 0 \;\Rightarrow\; H(x) = A\cos kx + B\sin kx$$

$$\frac{d^2Q}{dy^2} + p^2Q = 0 \;\Rightarrow\; Q(y) = C\cos py + D\sin py$$



Boundary Conditions:

$$u(0,y) = u(a,y) = 0$$
$$u(x,0) = u(x,b) = 0$$

$$\Rightarrow\; u(x,y,t) = H(x)Q(y)G(t)$$

$$H(0) = 0, \quad H(a) = 0, \quad Q(0) = 0, \quad Q(b) = 0.$$

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Partial Differential Equations (PDEs)

$$H(x) = A\cos kx + B\sin kx \qquad Q(y) = C\cos py + D\sin py$$

$$H(0) = 0, \quad H(a) = 0,$$

$$H(0) = A = 0$$
$$H(a) = B\sin ka = 0$$

$$\Rightarrow\; k = \frac{m\pi}{a} \quad (m \text{ integer}).$$

$$H_m(x) = \sin\frac{m\pi x}{a}$$

Similarly:

$$Q(0) = 0, \quad Q(b) = 0.$$

$$C = 0 \qquad p = \frac{n\pi y}{b} \;\Rightarrow\; Q_n(y) = \sin\frac{n\pi y}{b},$$

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Partial Differential Equations (PDEs)

We obtain:

$$(8)\quad F_{mn}(x, y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$n = 1, 2, \cdots, \qquad m = 1, 2, \cdots,$$

$$\left.\begin{array}{l} p^2 = \nu^2 - k^2 \\ \lambda = c\nu \end{array}\right\} \Rightarrow \lambda = c\sqrt{k^2 + p^2}.$$

$$\left.\begin{array}{l} k = m\pi/a \\ p = n\pi/b \end{array}\right\} \Rightarrow \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad (9)$$

Eigenvalues.

Partial Differential Equations (PDEs)

Recall "time function" ODE (equation (4)):

$$(4)\quad \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \qquad \ddot{G} + \lambda^2 G = 0$$

A corresponding general solution of (4) is

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.$$

$$(10)\quad u_{mn}(x, y, t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Partial Differential Equations (PDEs)

Step 3. Solution of the Model (1), (2), (3).
Double Fourier Series

To obtain the solutions that also satisfies initial conditions (3), we proceed as:

$$(14)\qquad u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

setting $t = 0$,

$$(15)\qquad u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y).$$

(15) is called the **double Fourier series** of $f(x, y)$

Partial Differential Equations (PDEs)

coefficients of **double Fourier** can be determined as follows:

Let:

$$(16)\qquad K_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}$$

we can write (15) in the form

$$f(x,y) = \sum_{m=1}^{\infty} K_m(y)\sin\frac{m\pi x}{a}.$$

For fixed y this is the Fourier sine series of $f(x, y)$, considered as a function of x.

$$(17)\qquad K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx.$$

Partial Differential Equations (PDEs)

Furthermore, (16) is the Fourier sine series of $K_m(y)$,

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\sin\frac{n\pi y}{b}\,dy.$$

From this and (17) we obtain the **generalized Euler formula**

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \qquad (18)$$

Partial Differential Equations (PDEs)

To determine the B^*_{mn},

differentiate (14) termwise with respect to t; using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y).$$

Then, proceeding as before, we find that the coefficients are

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy \qquad (19)$$

Solution Completed…

Partial Differential Equations (PDEs)

EXAMPLE 2 Vibration of a Rectangular Membrane

Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft (Fig. 305) if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft^2 (as for light rubber), the initial velocity is 0, and the initial displacement is

(20) $$f(x, y) = 0.1(4x - x^2)(2y - y^2) \text{ ft.}$$

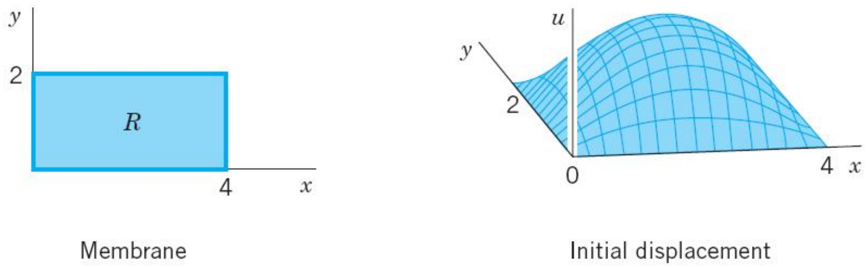


Fig. 305. Example 2

Partial Differential Equations (PDEs)

Solution. $c^2 = T/\rho = 12.5/2.5 = 5$ [ft^2/sec^2]. Also $B^*_{mn} = 0$ from (19). From (18) and (20),

$$B_{mn} = \frac{4}{4 \cdot 2} \int_0^2 \int_0^4 0.1(4x - x^2)(2y - y^2) \sin \frac{m\pi x}{4} \sin \frac{n\pi y}{2} \, dx \, dy$$

$$= \frac{1}{20} \int_0^4 (4x - x^2) \sin \frac{m\pi x}{4} dx \int_0^2 (2y - y^2) \sin \frac{n\pi y}{2} dy.$$

Two integrations by parts give for the first integral on the right

$$\frac{128}{m^3\pi^3} [1 - (-1)^m] = \frac{256}{m^3\pi^3} \qquad (m \text{ odd})$$

and for the second integral

For even m or n we get 0.

$$\frac{16}{n^3\pi^3} [1 - (-1)^n] = \frac{32}{n^3\pi^3} \qquad (n \text{ odd}).$$

Partial Differential Equations (PDEs)

Then we obtain:

$$B_{mn} = \frac{256 \cdot 32}{20 m^3 n^3 \pi^6} \approx \frac{0.426050}{m^3 n^3} \qquad (m \text{ and } n \text{ both odd}).$$

From this, (9), and (14) we obtain the answer

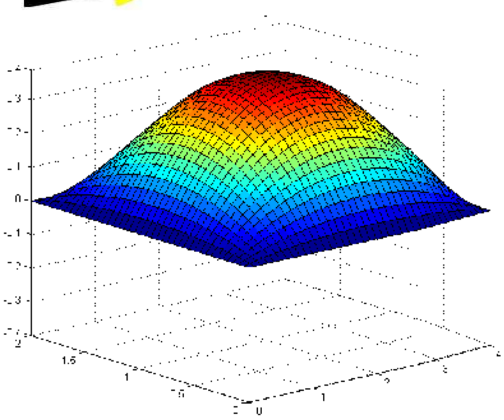
$$u(x, y, t) = 0.426050 \sum_{m,n \text{ odd}} \sum \frac{1}{m^3 n^3} \cos\left(\frac{\sqrt{5}\pi}{4} \sqrt{m^2 + 4n^2}\right) t \sin \frac{m\pi x}{4} \sin \frac{n\pi y}{2}$$

$$(21) \qquad = 0.426050 \left(\cos \frac{\sqrt{5}\pi\sqrt{5}}{4} t \sin \frac{\pi x}{4} \sin \frac{\pi y}{2} + \frac{1}{27} \cos \frac{\sqrt{5}\pi\sqrt{37}}{4} t \sin \frac{\pi x}{4} \sin \frac{3\pi y}{2} + \frac{1}{27} \cos \frac{\sqrt{5}\pi\sqrt{13}}{4} t \sin \frac{3\pi x}{4} \sin \frac{\pi y}{2} + \frac{1}{729} \cos \frac{\sqrt{5}\pi\sqrt{45}}{4} t \sin \frac{3\pi x}{4} \sin \frac{3\pi y}{2} + \cdots \right).$$

Partial Differential Equations (PDEs)

MATLAB

The Language of Technical Computing



Questions? Discussion? Suggestions ?

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