In the name of God

Engineering Mathematics

(Lecture # 12)

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Complex Analysis

Complex Analysis

Complex Analysis

COMPLEX ANALYSIS:

Complex analysis, traditionally known as the **theory of functions of a complex variable**, is the branch of mathematics that investigates functions of complex numbers. It is useful in many branches of mathematics, including number theory and applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, and off course **electrical engineering**.

Color plot of complex function : $f(x) = (x^2-1)(x-2-i)^2 / (x^2+2+2i)$, hue represents the argument, sat and value represents the modulus

Complex Analysis

HISTORY

Complex analysis is one of the classical branches in mathematics with roots in the 19th century and just prior. Important names are Euler, Gauss, Riemann, Cauchy, Weierstrass, and many more in the 20th century. Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory.

Complex Analysis

Topics:

- Complex Numbers and Functions. Complex Differentiation
- Complex Integration
- Power Series, Taylor Series
- Laurent Series. Residue Integration
- Conformal Mapping
- Complex Analysis and Potential Theory

Complex Analysis

Complex Numbers and Functions.

Motivation:

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$

Natural Numbers	Integer Numbers	Real Numbers
1, 2, 3, …	…, -2, -1, 0, 1, 2, …	…, -2.12,…, -1.5,…, 0, …, 1.02, …

Equations without *real* solutions, such as

$$x^2 = -1 \qquad x^2 - 10x + 40 = 0,$$

$$\mathbb{R} \subset \mathbb{C}$$

Italian mathematician GIROLAMO CARDANO (1501–1576),

CARL FRIEDRICH GAUSS

Complex Analysis

By definition,

a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

x is called the **real part** and y the **imaginary part** of z, written

$$x = \text{Re } z, \qquad y = \text{Im } z.$$

two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

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$(0, 1)$ is called the **imaginary unit** and is denoted by i,

(1) $$i = (0, 1).$$

Addition, Multiplication.

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

(2) $$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \quad y_1 + y_2).$$

Multiplication is defined by

(3) $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, \quad x_1 y_2 + x_2 y_1).$$

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the complex numbers "***extend***" the real numbers. We can thus write

$(x, 0) = x.$ Similarly, $(0, y) = iy$

In practice, complex numbers $z = (x, y)$ are written

$$z = x + iy \tag{4}$$

Remark:

Electrical engineers often write j instead of i because they need i for the current.

Complex Analysis

If $x = 0$, then $z = iy$ and is called **pure imaginary**. Also, (1) and (3) give

$$i^2 = -1 \tag{5}$$

because, by the definition of multiplication,

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1.$$

For **addition** the standard notation (4) gives

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Complex Analysis

Subtraction, Division

Subtraction and division are defined as the inverse operations of addition and multiplication, respectively.

$z = z_1 - z_2$ → $z_1 = z + z_2.$

(6) $$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

$z = z_1/z_2 \; (z_2 \neq 0)$ → $z_1 = zz_2.$

(7) $$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

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EXAMPLE

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$

$$z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$$

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}i.$$

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Complex Plane

Cartesian coordinate system.

$$z = x + iy$$

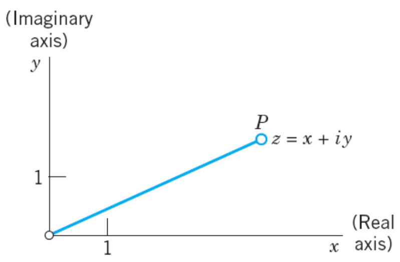

Fig. 318. The complex plane

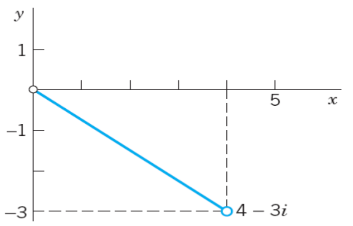

Fig. 319. The number 4 – 3i in the complex plane

Complex Analysis

Addition

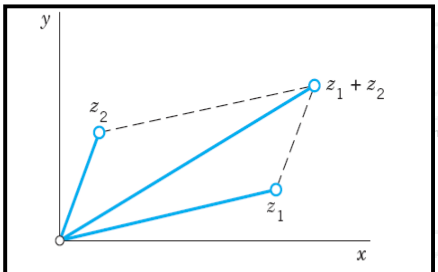

Fig. 320. Addition of complex numbers

Subtraction

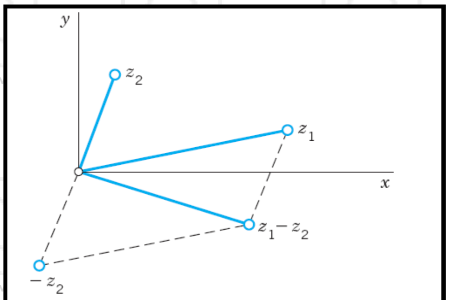

Fig. 321. Subtraction of complex numbers

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Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

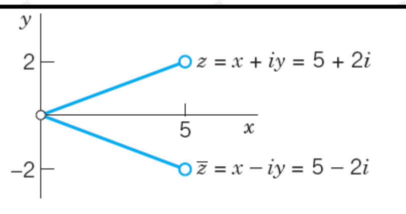

Fig. 322. Complex conjugate numbers

It is obtained geometrically by reflecting the point z in the real axis.

Complex Analysis

Important Formulas:

(8) $$\text{Re}\, z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \text{Im}\, z = y = \frac{1}{2i}(z - \bar{z}).$$

$$z\bar{z} = x^2 + y^2$$

(9) $$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

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EXAMPLE **Illustration of (8) and (9)**

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\operatorname{Im} z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$

$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

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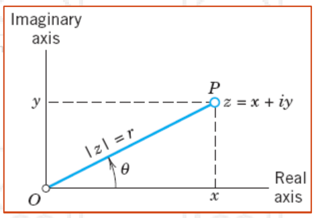
Polar Form of Complex Numbers.

(1) $x = r\cos\theta, \qquad y = r\sin\theta.$

We see that then $z = x + iy$ takes the so-called **polar form**

(2) $z = r(\cos\theta + i\sin\theta).$



r is called the **absolute value** or **modulus** of z and is denoted by $|z|$.

(3) $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$

Complex Analysis

θ is called the **argument** of z and is denoted by arg z. Thus $\theta = \arg z$ and

$$\tan\theta = \frac{y}{x} \quad (4)$$

in calculus, all *angles are measured in radians and positive in the counterclockwise sense*.

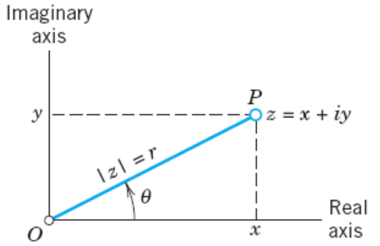


Fig. 323. Complex plane, polar form of a complex number

Complex Analysis

the **principal value** Arg z (with capital A!) of arg z by the double inequality

$$-\pi < \operatorname{Arg} z \leqq \pi. \quad (5)$$

$$\arg z = \operatorname{Arg} z \pm 2n\pi \quad (n = \pm 1, \pm 2, \cdots).$$

EXAMPLE 1 Polar Form of Complex Numbers. Principal Value Arg z

$z = 1 + i$ (Fig. 325) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$.

$|z| = \sqrt{2}, \quad \arg z = \frac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots),$

$\operatorname{Arg} z = \frac{1}{4}\pi \quad$ (the principal value).

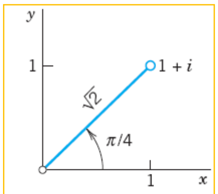


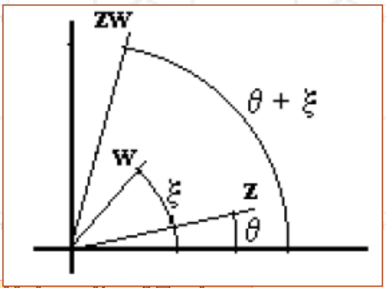
Fig. 325. Example 1

Complex Analysis

Example. Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\xi + i\sin\xi)$.

Then

$$zw = r(\cos\theta + i\sin\theta)s(\cos\xi + i\sin\xi)$$
$$= rs[(\cos\theta\cos\xi - \sin\theta\sin\xi) + i(\sin\theta\cos\xi + \sin\xi\cos\theta)]$$
$$= rs(\cos(\theta+\xi) + i\sin(\theta+\xi))$$



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Triangle Inequality

The daily bread of the complex analyst is the **triangle inequality**

$$|z_1 + z_2| \leqq |z_1| + |z_2| \quad (6)$$

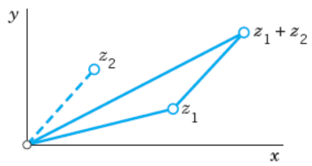

Fig. 326. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \quad (6^*)$$

Complex Analysis

EXAMPLE 2 **Triangle Inequality**

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$

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Multiplication and Division in Polar Form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Multiplication.

(7) $$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

(8) $$|z_1 z_2| = |z_1||z_2|.$$

(9) $$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

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Division.

(10) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

division by $|z_2|$

Note:

$z_1 = (z_1/z_2)z_2.$ → $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$

Similarly,

(11) $$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

(12) $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

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EXAMPLE 4 Integer Powers of z. De Moivre's Formula

with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

(13) $$z^n = r^n(\cos n\theta + i\sin n\theta).$$

For $|z| = r = 1$, formula (13) becomes

(13*) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

De Moivre's formula

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for $n = 2$ we have

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + \sin^2\theta$$
$$= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$
$$= \cos 2\theta + i\sin 2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\cos\theta\sin\theta$$

Complex Analysis

Roots of Complex Numbers:

If $z = w^n$ $(n = 1, 2, \cdots)$, then **nth root** of z,

$$w = \sqrt[n]{z}. \quad (14)$$

We write z and w in polar form

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad w = R(\cos\phi + i\sin\phi).$$

$$w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta).$$

$$R = \sqrt[n]{r},$$

$$n\phi = \theta + 2k\pi, \quad \text{thus} \quad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

Complex Analysis

Roots of Complex Numbers:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \tag{15}$$

multivalued, namely, *n-valued*.

Complex Analysis

Example:

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n-1. \tag{16}$$

These n values are called the **nth roots of unity**. They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 327–329 show $\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.

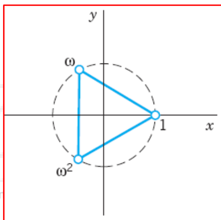

Fig. 327. $\sqrt[3]{1}$

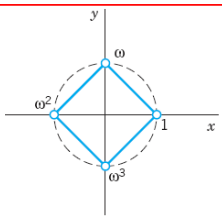

Fig. 328. $\sqrt[4]{1}$

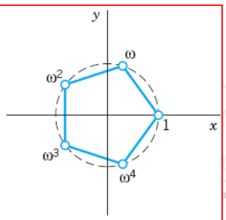

Fig. 329. $\sqrt[5]{1}$

Questions? Discussion? Suggestions ?

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