*In the name of God*

# Engineering Mathematics

## (Lecture # 13)

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## Complex Analysis

# Complex Analysis

## Complex Analysis

### Examples:

$(a)\ (1+i)^{12}=?\qquad (b)\ \sqrt[3]{3+4i}=?$

### Solution:

$$(a)\ (1+i)^{12}=\left((1+i)^2\right)^6=(1+2i-1)^6=(2i)^6=2^6 i^6$$
$$=2^6\left(i^2\right)^3=2^6(-1)^3=-2^6$$

$$\left|(1+i)^{12}\right|=2^6,\qquad Arg\left[(1+i)^{12}\right]=\pi$$



## Complex Analysis

$$(b)\ \sqrt[3]{3+4i}=r\left(\cos\left(\frac{\theta+2k\pi}{3}\right)+i\sin\left(\frac{\theta+2k\pi}{3}\right)\right)$$

$$\begin{cases} r=\sqrt[3]{|3+4i|}=\sqrt[3]{\left(3^2+4^2\right)^{\frac{1}{2}}}=\sqrt[3]{5} \\ \theta=Arg(3+4i)=\tan^{-1}\left(\frac{4}{3}\right)\end{cases}$$

$$\sqrt[3]{3+4i}=\sqrt[3]{5}\left(\cos\left(\frac{\theta}{3}\right)+i\sin\left(\frac{\theta}{3}\right)\right)$$
$$=\sqrt[3]{5}\left(\cos\left(\frac{\theta+2\pi}{3}\right)+i\sin\left(\frac{\theta+2\pi}{3}\right)\right)$$
$$=\sqrt[3]{5}\left(\cos\left(\frac{\theta+4\pi}{3}\right)+i\sin\left(\frac{\theta+4\pi}{3}\right)\right)$$

## Complex Analysis

### Example:

$$(a)\ z^4+(5-14i)z^2-(24+10i)=0$$

$$(b)\ z^4+16=0$$

### Solution:

$$(a)\ z^2=\frac{-(5-14i)\pm\sqrt{(5-14i)^2-4(-24-10i)}}{2}=\frac{-(5-14i)\pm\sqrt{-(75+100i)}}{2}$$

$$\sqrt{-(75+100i)}=\sqrt{125}\left(\cos\left(\frac{\theta_1}{2}\right)+i\sin\left(\frac{\theta_1}{2}\right)\right),\quad \theta_1=\tan^{-1}\left(\frac{-100}{-75}\right)$$

$$=\sqrt{125}\left(\cos\left(\frac{\theta_1+2\pi}{2}\right)+i\sin\left(\frac{\theta_1+2\pi}{2}\right)\right),$$

$$\boxed{\sqrt{-(75+100i)}=\{5-10i,\quad -5+10i\}}$$

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## Complex Analysis

$$\boxed{\sqrt{-(75+100i)}=\{5-10i,\quad -5+10i\}}$$

$$(a)\ z^2=\frac{-(5-14i)\pm\sqrt{-(75+100i)}}{2}=\begin{cases}\dfrac{-(5-14i)\pm(5-10i)}{2}\\[2ex]\dfrac{-(5-14i)\pm(-5+10i)}{2}\end{cases}$$

$$(a)\ z^2=\begin{cases}\dfrac{4i}{2}=2i\\[2ex]\dfrac{-10+24i}{2}=-5+12i\\[2ex]\dfrac{-10+24i}{2}=-5+12i\\[2ex]\dfrac{4i}{2}=2i\end{cases}\quad\Rightarrow\quad z=\begin{cases}\sqrt{2i}\\ \sqrt{-5+12i}\end{cases}$$

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$$z=\sqrt{2i}=\begin{cases}\sqrt{2}\left(\cos\left(\frac{\pi/2}{2}\right)+i\sin\left(\frac{\pi/2}{2}\right)\right)=1+i \rightarrow Z_1\\ \sqrt{2}\left(\cos\left(\frac{\pi/2+2\pi}{2}\right)+i\sin\left(\frac{\pi/2+2\pi}{2}\right)\right)=-1-i \rightarrow Z_2\end{cases}$$

$$z=\sqrt{-5+12i}=\begin{cases}\sqrt{13}\left(\cos\left(\frac{\tan^{-1}(12/-5)}{2}\right)+i\sin\left(\frac{\tan^{-1}(12/-5)}{2}\right)\right)=2+3i \rightarrow Z_3\\ \sqrt{13}\left(\cos\left(\frac{\tan^{-1}(12/-5)+2\pi}{2}\right)+i\sin\left(\frac{\tan^{-1}(12/-5)+2\pi}{2}\right)\right)=-2-3i \rightarrow Z_4\end{cases}$$



## Complex Analysis

$(b)\ z^4+16=0$

$z^4=-16 \rightarrow z=\sqrt[4]{-16}$

$$z=\sqrt[n]{r}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

$$z=\begin{cases}\sqrt[4]{16}\left(\cos\left(\frac{\pi+0}{4}\right)+i\sin\left(\frac{\pi+0}{4}\right)\right)=2\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=\sqrt{2}+i\sqrt{2},\\ \sqrt[4]{16}\left(\cos\left(\frac{\pi+2\pi}{4}\right)+i\sin\left(\frac{\pi+2\pi}{4}\right)\right)=2\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=-\sqrt{2}+i\sqrt{2},\\ \sqrt[4]{16}\left(\cos\left(\frac{\pi+4\pi}{4}\right)+i\sin\left(\frac{\pi+4\pi}{4}\right)\right)=2\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-\sqrt{2}-i\sqrt{2},\\ \sqrt[4]{16}\left(\cos\left(\frac{\pi+6\pi}{4}\right)+i\sin\left(\frac{\pi+6\pi}{4}\right)\right)=2\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=\sqrt{2}-i\sqrt{2},\end{cases}$$

## Complex Analysis

# Derivative. Analytic Function

## Circles

circular disk

unit circle $|z| = 1$

$|z - a| = \rho$

$\rho_1 \le |z - a| \le \rho_2$

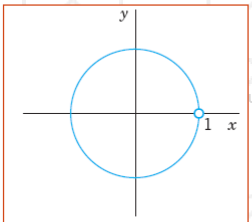



Fig. 330. Unit circle

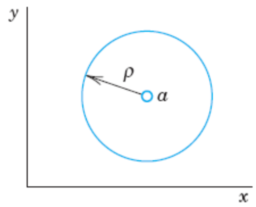



Fig. 331. Circle in the complex plane

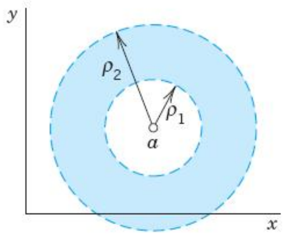
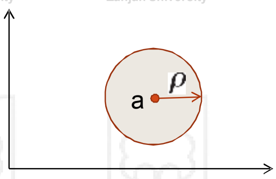



Fig. 332. Annulus in the complex plane



## Complex Analysis

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of $a$ or, more precisely, a *ρ-neighborhood* of $a$. And $a$ has infinitely many of them, one for each value of $\rho$ ($> 0$), and $a$ is a point of each of them, by definition!




**Half-Planes.** By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

## Complex Analysis

### Complex Function

$$w = f(z) = u(x, y) + iv(x, y).$$

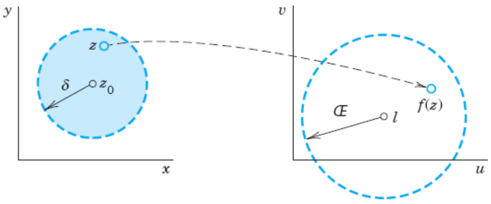



Fig. 333. Limit

### Limit, Continuity

A function $f(z)$ is said to have the **limit** $l$ as $z$ approaches a point $z_0$, written

(1) $$\lim_{z \to z_0} f(z) = l,$$

if $f$ is defined in a neighborhood of $z_0$ (except perhaps at $z_0$ itself) and if the values of $f$ are "close" to $l$ for all $z$ "close" to $z_0$; in precise terms, if for every positive real $\epsilon$ we can find a positive real $\delta$ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$ (Fig. 333) we have

(2) $$|f(z) - l| < \epsilon;$$

geometrically, if for every $z \neq z_0$ in that $\delta$-disk the value of $f$ lies in the disk (2).

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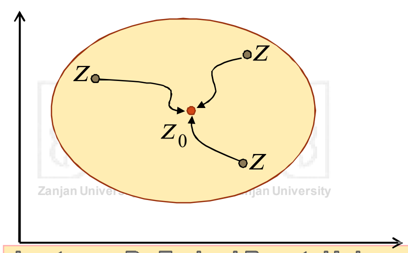
A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is defined and

(3) $$\lim_{z \to z_0} f(z) = f(z_0).$$

$f(z)$ is said to be *continuous in a domain* if it is continuous at each point of this domain.

### Remark:

$z$ may approach $z_0$ ***from any direction*** in the complex plane.

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## Complex Analysis

### Derivative

The **derivative** of a complex function $f$ at a point $z_0$ is written $f'(z_0)$ and is defined by

(4) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

(4′) $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Now comes an ***important point***. Remember that, by the definition of limit, $f(z)$ is defined in a neighborhood of $z_0$ and $z$ in (4′) may approach $z_0$ from any direction in the complex plane. Hence differentiability at $z_0$ means that, along whatever path $z$ approaches $z_0$, the quotient in (4′) always approaches a certain value and all these values are equal. This is important and should be kept in mind.



## Complex Analysis

**EXAMPLE Differentiability. Derivative**

$$f(z) = z^2$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

*The **differentiation rules** are the same as in real calculus,*

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

## Complex Analysis

**EXAMPLE** $\bar{z}$ **not Differentiable**

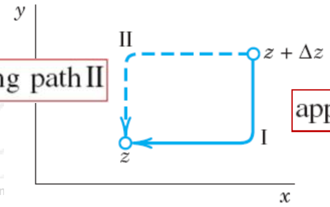
$$f(z) = \bar{z}$$

$$\Delta z = \Delta x + i\Delta y$$

$$(5) \qquad \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)}-\bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

the limit of (5) as $\Delta z \to 0$ does not exist at any $z$.

approaches $-1$ along path II




approaches $+1$ along path I

**Fig. 334.** Paths in (5)



## Complex Analysis

**Question**

$$f(z) = \mathrm{Re}(z)$$

# Analytic?

# Answer: NO!

# Questions? Discussion? Suggestions ?



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