دانشگاه زنجان

In the name of God

Engineering Mathematics

(Lecture # 14)

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Complex Analysis

Complex Analysis

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Question

$$f(z) = \bar{z}$$

$$f(z) = \mathrm{Re}(z)$$

Analytic?

Answer: NO!

Analytic Functions

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

A more modern term for *analytic in D* is *holomorphic in D*.

Complex Analysis

Polynomials, Rational Functions

polynomials, $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$

rational function. $f(z) = \dfrac{g(z)}{h(z)},$

Notes:

polynomials are analytic in the entire complex plane.

Rational functions are analytic in the entire complex plane, except where $h(z)=0$.

Complex Analysis

Cauchy–Riemann Equations.

The Cauchy–Riemann equations are the most important equations in this chapter

They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

THEOREM 1

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

(1) $$u_x = v_y, \qquad u_y = -v_x$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives.

Complex Analysis

THEOREM 2

Cauchy–Riemann Equations

*If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have **continuous** first partial derivatives that satisfy the Cauchy–Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.*

EXAMPLE

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

We have $u = e^x \cos y, v = e^x \sin y$

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$

$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

$f(z)$ is analytic for all z.

Complex Analysis

EXAMPLE **An Analytic Function of Constant Absolute Value Is Constant**

show that if $f(z)$ is analytic in a domain D and $|f(z)| = k = \text{const}$ in D, then $f(z) = \text{const}$ in D

Solution.

By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2.$

By differentiation,

$$uu_x + vv_x = 0,$$

$$uu_y + vv_y = 0.$$

Now use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, to get

$$\text{(6)} \qquad \text{(a)} \quad uu_x - vu_y = 0, \qquad \text{(b)} \quad uu_y + vu_x = 0.$$

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$$(6)\quad \begin{aligned} &(a) \quad uu_x - vu_y = 0, \\ &(b) \quad uu_y + vu_x = 0. \end{aligned}$$

multiply (6a) by u and (6b) by v and add.

multiply (6a) by $-v$ and (6b) by u and add.

$$(u^2 + v^2)u_x = 0,$$

$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $v_y = -v_x = 0$ Together this implies $u = \text{const}$ and $v = \text{const}$; hence $f = \text{const}$.

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We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are

$$(7)\quad \begin{aligned} u_r &= \frac{1}{r} v_\theta, \\ v_r &= -\frac{1}{r} u_\theta \end{aligned} \qquad (r > 0).$$

Proof it?

Complex Analysis

THEOREM

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{8}$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0, \tag{9}$$

in D and have continuous second partial derivatives in D.

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THEOREM

Laplace's Equation

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in D and have continuous second partial derivatives in D.

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory**

the real and imaginary parts of an analytic function are harmonic functions.

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If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. by direct calculation → $\nabla^2 u = 0$

$u_x = 2x$ → $v_y = u_x = 2x,$ → $v = 2xy + h(x),$

$u_y = -2y - 1$ → $v_x = -u_y = 2y + 1.$

$v_x = 2y + \frac{dh}{dx}$ → $dh/dx = 1$ → $h(x) = x + c.$

Complex Analysis

→ $v = 2xy + x + c$

The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$

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Question:
Determine *a so that the given function is* harmonic and find a harmonic conjugate.

$$u = e^{px}\cos(ay)$$

$$\left.\begin{aligned} u_{xx} &= +p^2 e^{px}\cos(ay) \\ u_{yy} &= -a^2 e^{px}\cos(ay) \end{aligned}\right\} \Rightarrow \boxed{IF \quad a = \pm p} \Rightarrow \boxed{\nabla^2 u = 0}$$

$$v_y = u_x = pe^{px}\cos(py) \Rightarrow v = e^{px}\sin(py) + h(x) \Rightarrow v_x = pe^{px}\sin(py) + \frac{dh(x)}{dx}$$

$$\Downarrow$$

$$\frac{dh(x)}{dx} = 0 \Leftarrow pe^{px}\sin(py) + \frac{dh(x)}{dx} = pe^{px}\sin(py) \Leftarrow v_x = -u_y$$

$$h(x) = c \Rightarrow \boxed{f(x,y) = e^{px}\cos(py) + i\left(e^{px}\sin(py) + c\right)}$$

Complex Analysis

Exponential Function

(1) $$e^z = e^x(\cos y + i \sin y).$$

This definition is motivated by the fact the e^z ***extends*** the real exponential function e^x of calculus in a natural fashion. Namely:

(A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.

(B) e^z is analytic for all z.

(C) The derivative of e^z is e^z, that is,

(2) $$(e^z)' = e^z.$$

Complex Analysis

Further Properties.

(3) $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

(4) $$e^z = e^x e^{iy}.$$

Euler formula

(5) $$e^{iy} = \cos y + i \sin y.$$

polar form

(6) $$z = re^{i\theta}.$$

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important formulas (verify!)

(7) $$e^{2\pi i} = 1$$

as well as

(8) $$e^{\pi i/2} = i, \quad e^{\pi i} = -1, \quad e^{-\pi i/2} = -i, \quad e^{-\pi i} = -1.$$

(9) $$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1.$$

(10) $$|e^z| = e^x. \quad \text{Hence} \quad \arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2, \cdots),$$

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Periodicity of e^z with period $2\pi i$,

(12) $$e^{z+2\pi i} = e^z \qquad \text{for all } z$$

fundamental region of e^z

(13) $$-\pi < y \leqq \pi$$

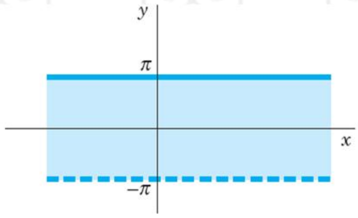

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

Complex Analysis

Trigonometric and Hyperbolic Functions.

(1) $$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Euler's formula ***is valid in complex***:

(5) $$e^{iz} = \cos z + i \sin z$$

Complex Analysis

EXAMPLE

Solve (a) $\cos z = 5$ (which has no real solution!),

Solution. From (1):

(1) $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 5 \;\Rightarrow\; e^{2iz} - 10e^{iz} + 1 = 0$$

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \quad \text{and} \quad 0.101.$$

$e^{-y} = 9.899$ or 0.101, $\Rightarrow$ $y = \pm 2.292$,

$e^{ix} = 1$, $\Rightarrow$ $x = 2n\pi$

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Questions? Discussion? Suggestions ?

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Complex Analysis

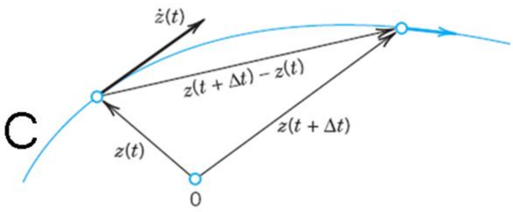
Complex Integration

Line Integral in the Complex Plane

$$\int_C f(z)\, dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve C or a portion of it

This curve C in the complex plane is called the **path of integration**.



First Evaluation Method: Indefinite Integration and Substitution of Limits

A domain D is called **simply connected** if every **simple closed curve** (closed curve without self-intersections) encloses only points of D.

THEOREM 1

Indefinite Integration of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

(9) $$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

Complex Analysis

Second Evaluation Method: Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function.

THEOREM 2

Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right). \tag{10}$$

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EXAMPLE 7 Integral of a Nonanalytic Function. Dependence on Path

Integrate $f(z) = \text{Re } z = x$ from 0 to $1 + 2i$ (a) along C^* in Fig. 343, (b) along C consisting of C_1 and C_2.

Solution.

(a)

C^* can be represented by $z(t) = t + 2it\ (0 \leqq t \leqq 1)$.

$\dot{z}(t) = 1 + 2i$

$f[z(t)] = x(t) = t$

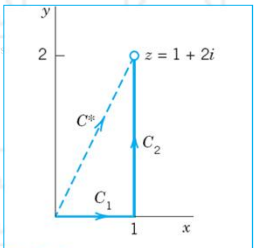


Fig. 343. Paths in Example 7

$$\int_{C^*} \text{Re } z\,dz = \int_0^1 t(1 + 2i)\,dt = \frac{1}{2}(1 + 2i) = \frac{1}{2} + i.$$

Complex Analysis

(b) We now have

$$C_1: z(t) = t, \qquad \dot{z}(t) = 1, \qquad f(z(t)) = x(t) = t \qquad (0 \leqq t \leqq 1)$$

$$C_2: z(t) = 1 + it, \qquad \dot{z}(t) = i, \qquad f(z(t)) = x(t) = 1 \qquad (0 \leqq t \leqq 2).$$

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} \operatorname{Re} z\, dz + \int_{C_2} \operatorname{Re} z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a).

Dependence on path. Now comes a very important fact. If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, ***a complex line integral depends not only on the endpoints of the path but in general also on the path itself***. The next example gives a first impression

Complex Analysis

Bounds for Integrals. *ML*-Inequality

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

$$\left| \int_C f(z)\, dz \right| \leqq ML \qquad \textbf{(13)} \quad (\textit{ML}\textbf{-inequality});$$

L is the length of C and M a constant such that $|f(z)| \leqq M$ everywhere on C.

PROOF

$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m \right| \leqq \sum_{m=1}^{n} |f(\zeta_m)|\, |\Delta z_m| \leqq M \sum_{m=1}^{n} |\Delta z_m|.$$

$|\Delta z_m|$ is the length of the chord whose endpoints are z_{m-1} and z_m (see Fig. 340).

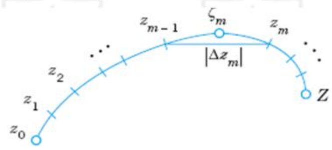


Fig. 340. Complex line integral

EXAMPLE 8 Estimation of an Integral

Find an upper bound for the absolute value of the integral $\int_C z^2\, dz$, C the straight-line segment from 0 to $1 + i$, Fig. 344.

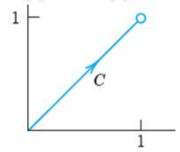


Fig. 344. Path in Example 8

Solution.

$$L = \sqrt{2}$$

$$|f(z)| = |z^2| \leqq 2 \text{ on } C$$

$$\Rightarrow \left| \int_C z^2\, dz \right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is

$$\left| -\tfrac{2}{3} + \tfrac{2}{3} i \right| = \tfrac{2}{3}\sqrt{2} = 0.9428$$

We cannot see from (13) how close to the bound ML the actual absolute value of the integral is

Cauchy's Integral Theorem

We have just seen that a line integral of a function f(z) generally depends not merely on the endpoints of the path, but also on the choice of the path itself. *However*, if f(z) is analytic in a domain *D and D is* simply connected, then the integral will not depend on the choice of a path between given points.

Hence:

conditions under which this path independence occurs are of considerable importance.

Complex Analysis

Let us continue our discussion of simple connectedness

1. A **simple closed path**

is a closed path that does not intersect or touch itself as shown in Fig. 345.

For example, a circle is simple, but a curve shaped like an 8 is not simple.

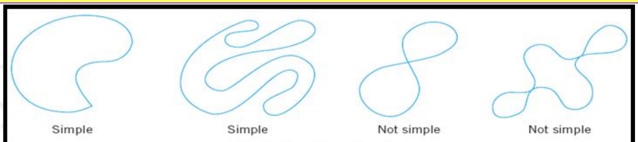


Fig. 345. Closed paths

2. A **simply connected domain** D in the complex plane

is a domain such that every simple closed path in D encloses only points of D.

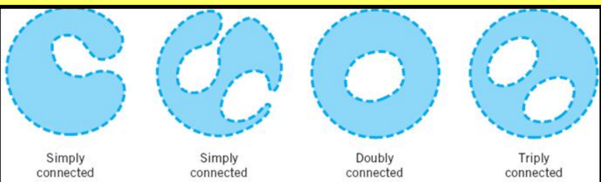


Fig. 346. Simply and multiply connected domains

Complex Analysis

THEOREM 1

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

(1) $$\oint_C f(z)\,dz = 0.$$ See Fig. 347.

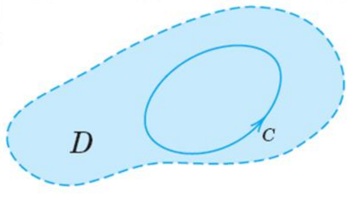


Fig. 347. Cauchy's integral theorem

Complex Analysis

EXAMPLE 1 Entire Functions

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all z).

EXAMPLE 2 Points Outside the Contour Where $f(x)$ is Not Analytic

$$\oint_C \sec z\,dz = 0, \qquad \oint_C \frac{dz}{z^2+4} = 0$$

where C is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C; none lies on C or inside C. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2i$ outside C.

Complex Analysis

EXAMPLE 3 Nonanalytic Function

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i$$

where C: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic.

EXAMPLE 4 Analyticity Sufficient, Not Necessary

$$\oint_C \frac{dz}{z^2} = 0$$

where C is the unit circle. This result does *not* follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence *the condition that f be analytic in D is sufficient rather than necessary for* (1) *to be true.*

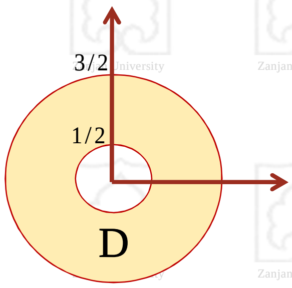
Complex Analysis

EXAMPLE 5 **Simple Connectedness Essential**

$$\oint_C \frac{dz}{z} = 2\pi i$$

for counterclockwise integration around the unit circle

C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $1/z$ is analytic, but this domain is not simply connected,



Complex Analysis

THEOREM 2

Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Existence of Indefinite Integral

THEOREM 3

Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D—thus, $F'(z) = f(z)$—which is analytic in D, and for all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by formula (9)

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]. \tag{9}$$

Complex Analysis

Cauchy's Integral Theorem for Multiply Connected Domains

doubly connected domain D with outer boundary curve C_1 and inner C_2 (Fig. 353).

If

a function f(z) is analytic in any domain D* that contains D *and its boundary curves,*

we claim that:

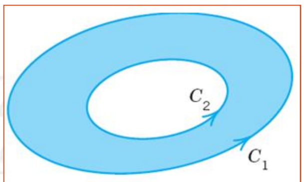


Fig. 353. Paths in (5)

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \qquad (6)$$

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PROOF

By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 354) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic.

By Cauchy's integral theorem

$$\oint_{D_1} f\,dz = 0 \qquad \oint_{D_2} f\,dz = 0$$

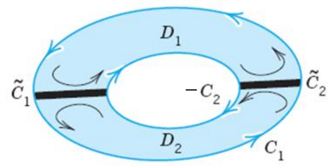

Fig. 354. Doubly connected domain

$$\oint_{D_1} fdz + \oint_{D_2} fdz = 0 \Rightarrow \oint_{C_1} fdz - \oint_{C_2} fdz = 0$$

integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions

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For domains of higher connectivity the idea remains the same.

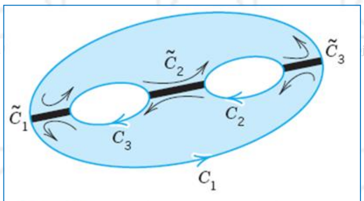

Fig. 355. Triply connected domain

Complex Analysis

Cauchy's Integral Formula

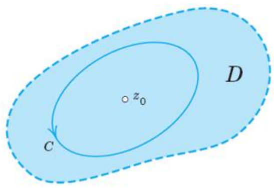

Fig. 356. Cauchy's integral formula

THEOREM 1

Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

(1) $$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$ **(Cauchy's integral formula)**

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

(1*) $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$ **(Cauchy's integral formula).**

Complex Analysis

EXAMPLE 1 **Cauchy's Integral Formula**

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i e^z \Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside

EXAMPLE 2 **Cauchy's Integral Formula**

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}\,dz$$

$$= 2\pi i[\tfrac{1}{2}z^3-3]|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i$$

$(z_0 = \frac{1}{2}i \text{ inside } C)$.

Complex Analysis

EXAMPLE 3 **Integration Around Different Contours**

Integrate

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$$

counterclockwise around each of the four circles in Fig. 358.

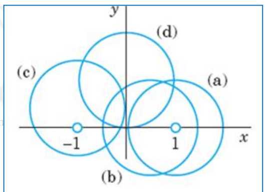

Fig. 358. Example 3

Solution.

$g(z)$ is not analytic at -1 and 1. These are the points we have to watch for.

(a) The circle $|z-1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1}; \quad \rightarrow \quad f(z) = \frac{z^2+1}{z+1} \quad \rightarrow$$

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

Complex Analysis

(b) gives the same as (a) by the principle of deformation of path.

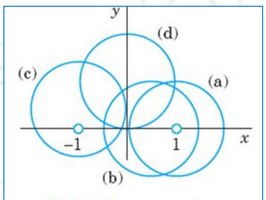

Fig. 358. Example 3

(c) The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1).

$$g(z) = \frac{z^2+1}{z-1}\,\frac{1}{z+1}; \quad \rightarrow \quad f(z) = \frac{z^2+1}{z-1}. \quad \rightarrow$$

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i \left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

(d) gives 0. Why?

Complex Analysis

Multiply connected domains can be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 (Fig. 359) and z_0 is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz, \tag{3}$$

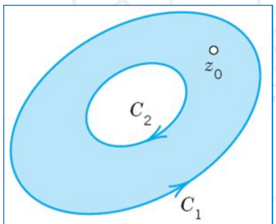

Fig. 359. Formula (3)

where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 359.

Complex Analysis

Derivatives of Analytic Functions

THEOREM 1

Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$(1') \qquad f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$(1'') \qquad f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

and in general

$$\textbf{(1)} \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots);$$

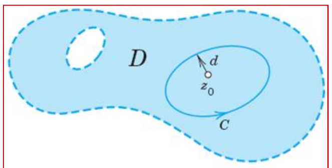

Fig. 360. Theorem 1 and its proof

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C (Fig. 360).

Complex Analysis

Applications of Theorem 1

EXAMPLE 1 **Evaluation of Line Integrals**

From $(1')$, for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi.$$

EXAMPLE 2 From $(1'')$, for any contour enclosing the point $-i$ we obtain by counterclockwise integration

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \pi i(z^4 - 3z^2 + 6)''\Big|_{z=-i} = \pi i[12z^2 - 6]_{z=-i} = -18\pi i.$$

EXAMPLE 3 By $(1')$, for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1}$$

$$= 2\pi i\,\frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Bigg|_{z=1} = \frac{6e\pi}{25}\,i \approx 2.050i.$$

Complex Analysis

Cauchy's Inequality.

$$|f^{(n)}(z_0)| \leqq \frac{n!M}{r^n}. \quad (2)$$

for C a circle of radius r and center z_0

with $|f(z)| \leqq M$ on C

Proof:

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r.$$

Complex Analysis

THEOREM 2

Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

THEOREM 3

Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz = 0 \quad (3)$$

for every closed path in D, then $f(z)$ is analytic in D.

SUMMARY OF CHAPTER

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The **complex line integral** of a function $f(z)$ taken over a path C is denoted by

$$(1) \qquad \int_C f(z)\,dz \qquad \text{or, if } C \text{ is closed, also by} \qquad \oint_C f(z)$$

If $f(z)$ is analytic in a simply connected domain D, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

$$(2) \qquad \int_C f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of C, where $a \leqq t \leqq b$,

$$(3) \qquad \int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Complex Analysis

Cauchy's integral theorem is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D (Sec. 14.2),

$$\oint_C f(z)\,dz = 0. \tag{4}$$

Under the same assumptions and for any z_0 in D and closed path C in D containing z_0 in its interior we also have **Cauchy's integral formula**

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{5}$$

Furthermore, under these assumptions $f(z)$ has derivatives of all orders in D that are themselves analytic functions in D and (Sec. 14.4)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots). \tag{6}$$

Questions? Discussion? Suggestions ?

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