In the name of God

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Engineering Mathematics

(Lecture # 15)

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Complex Analysis

Complex Analysis

Complex Analysis

General formulas

(9) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

(10) $\cos^2 z + \sin^2 z = 1.$

Hyperbolic Functions

(11) $\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}).$

Complex Trigonometric and Hyperbolic Functions Are Related.

(14) $\cosh iz = \cos z, \qquad \sinh iz = i \sin z.$

(15) $\cos iz = \cosh z, \qquad \sin iz = i \sinh z.$

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Complex Analysis

Logarithm. General Power. Principal Value

$w = \ln z$, $z \neq 0$ → $e^w = z.$

If we set $\begin{cases} w = u + iv \\ z = re^{i\theta}, \end{cases}$ → $e^w = e^{u+iv} = re^{i\theta}.$

$u = \ln r,$
$v = \theta.$

(1) $(r = |z| > 0, \quad \theta = \arg z).$ $\ln z = \ln r + i\theta$

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(1) $\ln z = \ln r + i\theta$

the complex natural logarithm* $\ln z$ $(z \neq 0)$ *is infinitely many-valued.

principal value of $\ln z$ $\qquad -\pi \leq Arg(z) \leq \pi$

(2) $(z \neq 0)$. $\qquad \text{Ln}\, z = \ln |z| + i\, \text{Arg}\, z$

(3) $\ln z = \text{Ln}\, z \pm 2n\pi i \quad (n = 1, 2, \cdots)$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm.

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(3) $\ln z = \text{Ln}\, z \pm 2n\pi i \quad (n = 1, 2, \cdots)$

(4a) $e^{\ln z} = z$

(4b) $\ln (e^z) = z \pm 2n\pi i,$

THEOREM 1

Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ *formula* (3) *defines a function, which is analytic, except at* 0 *and on the negative real axis, and has the derivative*

(6) $(\ln z)' = \dfrac{1}{z} \quad$ (z not 0 or negative real).

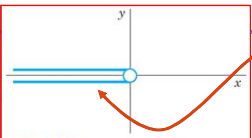

Fig. 338. Branch cut for ln z

Note: ln(z) is not continuous at negative real axis, because Arg(z) is not continuous there.

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PROOF We show that the Cauchy–Riemann equations are satisfied.

$$\ln z = \ln r + i(\theta + c) = \underbrace{\frac{1}{2}\ln(x^2 + y^2)}_{u(x,\,y)} + i\underbrace{\left(\arctan\frac{y}{x} + c\right)}_{v(x,\,y)}$$

where the constant c is a multiple of 2π.

Then:

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right).$$

Cauchy–Riemann equations hold.

Then:

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

Complex Analysis

General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

(7) $$z^c = e^{c \ln z} \quad (c \text{ complex}, z \neq 0).$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued.

The particular value $z^c = e^{c \operatorname{Ln} z}$

is called the **principal value** *of* z^c.

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z} \quad n \text{ distinct values}$$

EXAMPLE $i^i = ?$ $\qquad (1+i)^{2-i} = ?$

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right] = e^{-(\pi/2)\mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly,

$$(1+i)^{2-i} = \exp[(2-i)\ln(1+i)] = \exp[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}]$$

$$= 2e^{\pi/4 \pm 2n\pi}[\sin(\tfrac{1}{2}\ln 2) + i\cos(\tfrac{1}{2}\ln 2)].$$

SUMMARY OF CHAPTER

Complex Numbers and Functions.

Complex Differentiation

Complex Analysis

complex numbers

$$(1) \qquad z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$$r = |z| = \sqrt{x^2 + y^2},$$

$$\theta = \arctan(y/x),$$

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

$$(2) \qquad f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

Complex Analysis

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

$$(3) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in D.

Then u and v also satisfy **Laplace's equation**

$$(4) \qquad u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

everywhere in D.

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The complex **exponential function**

$$(5) \qquad e^z = \exp z = e^x (\cos y + i \sin y)$$

reduces to e^x if $z = x$ $(y = 0)$.

It is periodic with $2\pi i$ and has the derivative e^z.

The **natural logarithm** is

$$(8) \qquad \ln z = \ln|z| + i \arg z = \ln|z| + i \operatorname{Arg} z \pm 2n\pi i$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z,

$$-\pi < \operatorname{Arg} z \leqq \pi.$$

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Complex Integration

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Line Integral in the Complex Plane

$$\int_C f(z)\, dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve C or a portion of it

This curve C in the complex plane is called the **path of integration**.

We may represent C by a parametric representation

(1) $$z(t) = x(t) + iy(t) \quad (a \leqq t \leqq b).$$

For instance,

$z(t) = t + 3it \ (0 \leqq t \leqq 2)$ → (a segment) of the line $y = 3x$.

$z(t) = 4\cos t + 4i \sin t \ (-\pi \leqq t \leqq \pi)$ → circle $|z| = 4$,

We assume C to be a **smooth curve**, that is, C has a continuous and nonzero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}$$

Here we use a dot since a prime $'$ denotes the derivative with respect to z.

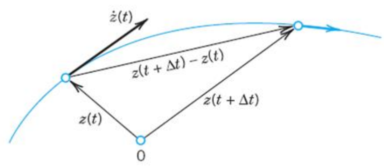

Fig. 339. Tangent vector $\dot{z}(t)$ of a curve C in the complex plane given by $z(t)$.

Complex Analysis

Definition of the Complex Line Integral

subdivide (we "***partition***") the interval $a \leqq t \leqq b$ in (1) by points

$$t_0\,(=a),\quad t_1,\quad \cdots,\quad t_{n-1},\quad t_n\,(=b)$$

where $t_0 < t_1 < \cdots < t_n$. To this subdivision there corresponds a subdivision of C by points

$$z_0,\quad z_1,\quad \cdots,\quad z_{n-1},\quad z_n\,(=Z)$$

where $z_j = z(t_j)$.

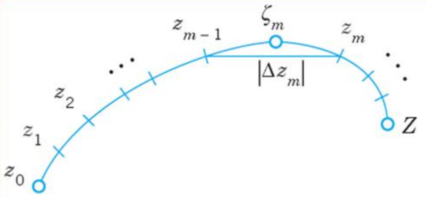

Fig. 340. Complex line integral

Complex Analysis

On each portion of subdivision of C we choose an arbitrary point,

ζ_1 between z_0 and z_1

a point ζ_2 between z_1 and z_2, etc. Then we form the sum

$$(2)\qquad S_n = \sum_{m=1}^{n} f(\zeta_m)\,\Delta z_m \qquad \text{where} \qquad \Delta z_m = z_m - z_{m-1}.$$

We do this for each $n = 2, 3, \cdots$ in a completely independent manner.

The limit of the sequence of complex numbers $S_2, S_3, \cdots$ thus obtained is called the **line integral**

$$(3)\qquad \int_C f(z)\,dz, \qquad \text{or by} \qquad \oint_C f(z)\,dz$$

Complex Analysis

Basic Properties Directly Implied by the Definition

1. Linearity.

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\, dz = k_1 \int_C f_1(z)\, dz + k_2 \int_C f_2(z)\, dz. \tag{4}$$

2. Sense reversal in integrating over the ***same*** path, from z_0 to Z (left) and from Z to z_0 (right), introduces a minus sign as shown,

$$\int_{z_0}^{Z} f(z)\, dz = -\int_{Z}^{z_0} f(z)\, dz. \tag{5}$$

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3. Partitioning of path

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz. \tag{6}$$

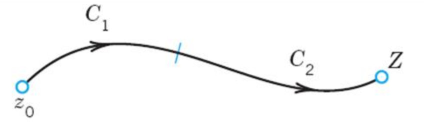

Fig. 341. Partitioning of path [formula (6)]

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First Evaluation Method: Indefinite Integration and Substitution of Limits

This method is the analog of the evaluation of definite integrals in calculus by the well-known formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where } [F'(x) = f(x)].$$

it is suitable for analytic functions only.

A domain D is called **simply connected** if every **simple closed curve** (closed curve without self-intersections) encloses only points of D.

THEOREM 1

Indefinite Integration of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

(9)
$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

EXAMPLE 1

$$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3}+\frac{2}{3}i$$

EXAMPLE 2

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

since e^z is periodic with period $2\pi i$.

EXAMPLE 3

$$\int_{-i}^{i} \frac{dz}{z} = \text{Ln}\,i - \text{Ln}\,(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi.$$

Here D is the complex plane without 0 and the negative real axis (where Ln z is not analytic). Obviously, D is a simply connected domain.

Second Evaluation Method: Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function.

THEOREM 2

Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

(10)
$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Complex Analysis

Steps in Applying Theorem 2

(A) Represent the path C in the form $z(t)$ $(a \leqq t \leqq b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

EXAMPLE 5 A Basic Result: Integral of 1/z Around the Unit Circle

We show that by integrating $1/z$ counterclockwise around the unit circle we obtain

$$\oint_C \frac{dz}{z} = 2\pi i \tag{11}$$

(C the unit circle, counterclockwise).

This is a very important result that we shall need quite often.

Complex Analysis

Solution. **(A)** We may represent the unit circle C

$$z(t) = \cos t + i \sin t = e^{it} \qquad (0 \leqq t \leqq 2\pi),$$

(B) Differentiation gives $\dot{z}(t) = ie^{it}$ (chain rule!).

(C) By substitution, $f(z(t)) = 1/z(t) = e^{-it}$.

(D) From (10) we thus obtain the result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

Remark:

Theorem 1 can not be applied for this case, because 1/z is not analytic at z=0

And there is not any simple connected D containing curve C (unit circle)!.

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EXAMPLE 6 **Integral of $1/z^m$ with Integer Power m**

Let $f(z) = (z - z_0)^m$ where m is the integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0 (Fig. 342).

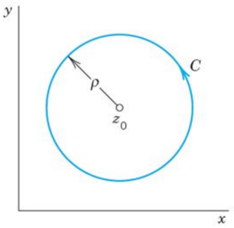

Fig. 342. Path in Example 6

Solution.

We may represent C in the form

$$z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \quad (0 \leqq t \leqq 2\pi).$$

$$(z - z_0)^m = \rho^m e^{imt}, \qquad dz = i\rho e^{it}\, dt$$

$$\oint_C (z - z_0)^m\, dz = \int_0^{2\pi} \rho^m e^{imt}\, i\rho e^{it}\, dt = i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt.$$

the right side equals

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos\,(m+1)t\, dt + i\int_0^{2\pi} \sin\,(m+1)t\, dt\right].$$

Complex Analysis

$$i\rho^{m+1}\left[\int_0^{2\pi} \cos\,(m+1)t\, dt + i\int_0^{2\pi} \sin\,(m+1)t\, dt\right].$$

If $m = -1$, we have $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. We thus obtain $2\pi i$.

For integer $m \neq -1$ each of the two integrals is zero

(12)
$$\oint_C (z - z_0)^m\, dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}). \end{cases}$$

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Questions? Discussion? Suggestions ?

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