In the name of God

Engineering Mathematics

(Lecture # 16)

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Complex Analysis

Complex Integration

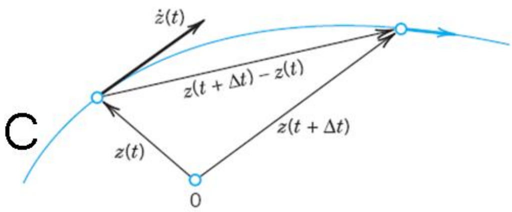
Complex Analysis

Line Integral in the Complex Plane

$$\int_C f(z)\,dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve C or a portion of it

This curve C in the complex plane is called the **path of integration**.



Complex Analysis

First Evaluation Method: Indefinite Integration and Substitution of Limits

A domain D is called **simply connected** if every **simple closed curve** (closed curve without self-intersections) encloses only points of D.

THEOREM 1

Indefinite Integration of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

(9) $$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

Second Evaluation Method: Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function.

THEOREM 2

Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right). \tag{10}$$

EXAMPLE 7 Integral of a Nonanalytic Function. Dependence on Path

Integrate $f(z) = \operatorname{Re} z = x$ from 0 to $1 + 2i$ (a) along C^* in Fig. 343, (b) along C consisting of C_1 and C_2.

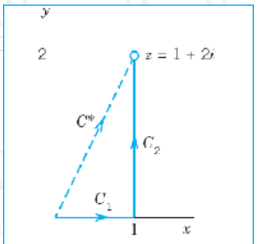


Fig. 343. Paths in Example 7

Solution.

(a)

C^* can be represented by $z(t) = t + 2it\ (0 \leqq t \leqq 1)$.

$\dot{z}(t) = 1 + 2i$

$f[z(t)] = x(t) = t$

$$\int_{C^*} \operatorname{Re} z\,dz = \int_0^1 t(1 + 2i)\,dt = \frac{1}{2}(1 + 2i) = \frac{1}{2} + i.$$

Complex Analysis

(b) We now have

$$C_1: z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad (0 \leqq t \leqq 1)$$

$$C_2: z(t) = 1 + it, \quad \dot{z}(t) = i, \quad f(z(t)) = x(t) = 1 \quad (0 \leqq t \leqq 2).$$

$$\int_C \operatorname{Re} z\, dz = \int_{C_1} \operatorname{Re} z\, dz + \int_{C_2} \operatorname{Re} z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a).

Dependence on path. Now comes a very important fact. If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, ***a complex line integral depends not only on the endpoints of the path but in general also on the path itself.*** The next example gives a first impression

Complex Analysis

Bounds for Integrals. *ML*-Inequality

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

$$\left|\int_C f(z)\, dz\right| \leqq ML \qquad (13) \quad (\textit{ML}\text{-inequality});$$

L is the length of C and M a constant such that $|f(z)| \leqq M$ everywhere on C.

PROOF

$$|S_n| = \left|\sum_{m=1}^{n} f(\zeta_m)\, \Delta z_m\right| \leqq \sum_{m=1}^{n} |f(\zeta_m)|\,|\Delta z_m| \leqq M \sum_{m=1}^{n} |\Delta z_m|.$$

$|\Delta z_m|$ is the length of the chord whose endpoints are z_{m-1} and z_m (see Fig. 340).

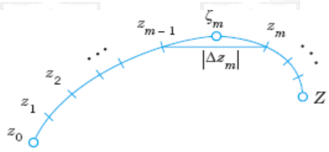


Fig. 340. Complex line integral

Complex Analysis

EXAMPLE 8 Estimation of an Integral

Find an upper bound for the absolute value of the integral $\int_C z^2\,dz$, C the straight-line segment from 0 to $1 + i$, Fig. 344.

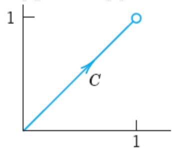


Fig. 344. Path in Example 8

Solution.

$L = \sqrt{2}$

$|f(z)| = |z^2| \leqq 2$ on C

$$\left|\int_C z^2\,dz\right| \leqq 2\sqrt{2} = 2.8284.$$

The absolute value of the integral is

$$\left|-\tfrac{2}{3} + \tfrac{2}{3}i\right| = \tfrac{2}{3}\sqrt{2} = 0.9428$$

We cannot see from (13) how close to the bound ML the actual absolute value of the integral is

Complex Analysis

Cauchy's Integral Theorem

We have just seen that a line integral of a function **f(z)** generally depends not merely on the **endpoints** of the path, **but also on the choice of the path** itself. ***However,*** if **f(z)** is analytic in a domain ***D*** *and* ***D*** *is* **simply connected**, then the integral will not depend on the choice of a path between given points.

Hence:

conditions under which this path independence occurs are of considerable importance.

Complex Analysis

Let us continue our discussion of simple connectedness

1. A **simple closed path**

is a closed path that does not intersect or touch itself as shown in Fig. 345.

For example, a circle is simple, but a curve shaped like an 8 is not simple.

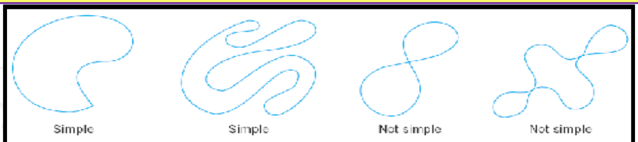


Fig. 345. Closed paths

2. A **simply connected domain** D in the complex plane

is a domain such that every simple closed path in D encloses only points of D.

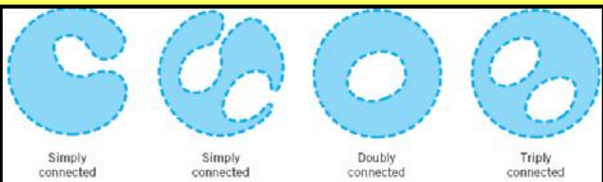


Fig. 346. Simply and multiply connected domains

Complex Analysis

THEOREM 1

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

(1)
$$\oint_C f(z)\,dz = 0.$$
See Fig. 347.

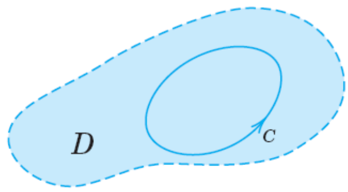


Fig. 347. Cauchy's integral theorem

Complex Analysis

EXAMPLE 1 Entire Functions

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all z).

EXAMPLE 2 Points Outside the Contour Where $f(x)$ is Not Analytic

$$\oint_C \sec z\,dz = 0, \qquad \oint_C \frac{dz}{z^2+4} = 0$$

where C is the unit circle, $\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside C; none lies on C or inside C. Similarly for the second integral, whose integrand is not analytic at $z = \pm 2i$ outside C.

Complex Analysis

EXAMPLE 3 Nonanalytic Function

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = 2\pi i$$

where C: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic.

EXAMPLE 4 Analyticity Sufficient, Not Necessary

$$\oint_C \frac{dz}{z^2} = 0$$

where C is the unit circle. This result does *not* follow from Cauchy's theorem, because $f(z) = 1/z^2$ is not analytic at $z = 0$. Hence *the condition that f be analytic in D is sufficient rather than necessary for* (1) *to be true.*

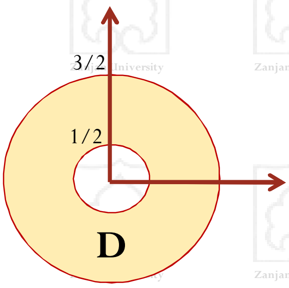
Complex Analysis

EXAMPLE 5 **Simple Connectedness Essential**

$$\oint_C \frac{dz}{z} = 2\pi i$$

for counterclockwise integration around the unit circle

C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $1/z$ is analytic, but this domain is not simply connected,



Complex Analysis

THEOREM 2

Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Existence of Indefinite Integral

THEOREM 3

Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D—thus, $F'(z) = f(z)$—which is analytic in D, and for all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by formula (9)

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \tag{9}$$

Complex Analysis

Cauchy's Integral Theorem for Multiply Connected Domains

doubly connected domain D with outer boundary curve C_1 and inner C_2 (Fig. 353).

If

a function f(z) is analytic in any domain D* that contains D

and its boundary curves,

we claim that:

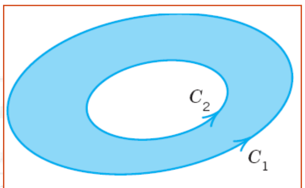


Fig. 353. Paths in (5)

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \qquad (6)$$

Complex Analysis

PROOF

By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 354) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic.

By Cauchy's integral theorem

$$\oint_{D_1} f\,dz = 0 \qquad \oint_{D_2} f\,dz = 0$$

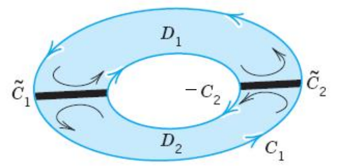


Fig. 354. Doubly connected domain

$$\oint_{D_1} fdz + \oint_{D_2} fdz = 0 \Rightarrow \oint_{C_1} fdz - \oint_{C_2} fdz = 0$$

integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions

Complex Analysis

For domains of higher connectivity the idea remains the same.

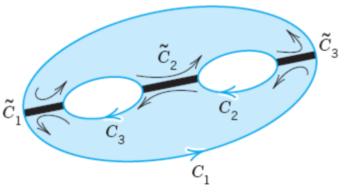


Fig. 355. Triply connected domain

Complex Analysis

Cauchy's Integral Formula

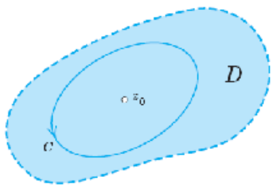


Fig. 356. Cauchy's integral formula

THEOREM 1

Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 356),

(1) $$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$ **(Cauchy's integral formula)**

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

(1*) $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$ **(Cauchy's integral formula).**

Complex Analysis

EXAMPLE 1 **Cauchy's Integral Formula**

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i e^z \Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside

EXAMPLE 2 **Cauchy's Integral Formula**

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}\,dz$$

$$= 2\pi i\left[\tfrac{1}{2}z^3 - 3\right]\big|_{z=i/2}$$

$$= \frac{\pi}{8} - 6\pi i$$

($z_0 = \frac{1}{2}i$ inside C).

Complex Analysis

EXAMPLE 3 **Integration Around Different Contours**

Integrate

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$$

counterclockwise around each of the four circles in Fig. 358.

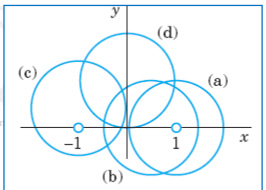


Fig. 358. Example 3

Solution.

$g(z)$ is not analytic at -1 and 1. These are the points we have to watch for.

(a) The circle $|z-1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1};\quad \Rightarrow \quad f(z) = \frac{z^2+1}{z+1} \quad \Rightarrow$$

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

Complex Analysis

(b) gives the same as (a) by the principle of deformation of path.

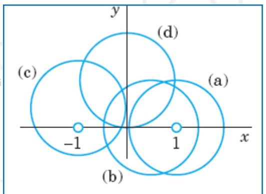


Fig. 358. Example 3

(c) The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1).

$g(z) = \frac{z^2+1}{z-1}\frac{1}{z+1}$; → $f(z) = \frac{z^2+1}{z-1}$. →

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i.$$

(d) gives 0. Why?

Complex Analysis

Multiply connected domains can be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 (Fig. 359) and z_0 is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz, \quad (3)$$

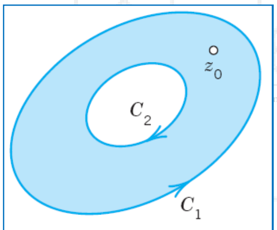


Fig. 359. Formula (3)

where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 359.

Complex Analysis

Derivatives of Analytic Functions

THEOREM 1

Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

(1′) $$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

(1″) $$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

and in general

(1) $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C (Fig. 360).

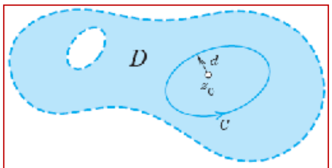


Fig. 360. Theorem 1 and its proof

Complex Analysis

Applications of Theorem 1

EXAMPLE 1 **Evaluation of Line Integrals**

From (1′), for any contour enclosing the point πi (counterclockwise)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi.$$

EXAMPLE 2 From (1″), for any contour enclosing the point $-i$ we obtain by counterclockwise integration

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \pi i(z^4 - 3z^2 + 6)''\Big|_{z=-i} = \pi i[12z^2 - 6]_{z=-i} = -18\pi i.$$

EXAMPLE 3 By (1′), for any contour for which 1 lies inside and $\pm 2i$ lie outside (counterclockwise),

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1}$$

$$= 2\pi i\,\frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2}\Big|_{z=1} = \frac{6e\pi}{25}\,i \approx 2.050i.$$

Complex Analysis

Cauchy's Inequality.

$$|f^{(n)}(z_0)| \leqq \frac{n!M}{r^n}. \qquad (2)$$

for C a circle of radius r and center z_0

with $|f(z)| \leqq M$ on C

Proof:

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leqq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r.$$

Complex Analysis

THEOREM 2

Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

THEOREM 3

Morera's[2] Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)\,dz = 0 \qquad (3)$$

for every closed path in D, then $f(z)$ is analytic in D.

Complex Analysis

SUMMARY OF CHAPTER

Complex Analysis

The **complex line integral** of a function $f(z)$ taken over a path C is denoted by

(1) $$\int_C f(z)\, dz \qquad \text{or, if } C \text{ is closed, also by} \qquad \oint_C f(z)$$

If $f(z)$ is analytic in a simply connected domain D, then we can evaluate (1) as in calculus by indefinite integration and substitution of limits, that is,

(2) $$\int_C f(z)\, dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)]$$

A general method of integration, not restricted to analytic functions, uses the equation $z = z(t)$ of C, where $a \leqq t \leqq b$,

(3) $$\int_C f(z)\, dz = \int_a^b f(z(t))\dot{z}(t)\, dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Complex Analysis

Cauchy's integral theorem is the most important theorem in this chapter. It states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D (Sec. 14.2),

$$\oint_C f(z)\,dz = 0. \tag{4}$$

Under the same assumptions and for any z_0 in D and closed path C in D containing z_0 in its interior we also have **Cauchy's integral formula**

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{5}$$

Furthermore, under these assumptions $f(z)$ has derivatives of all orders in D that are themselves analytic functions in D and (Sec. 14.4)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots). \tag{6}$$

Questions? **Discussion?** Suggestions ?