In the name of God

دانشگاه زنجان

# Engineering Mathematics

## (Lecture # 18)

**Instructor:**
Dr. Farhad Bayat

Zanjan University

**Email:** bayat.farhad@gmail.com

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## Complex Analysis



# Power Series, Taylor Series

## Power Series

A **power series** *in powers of* $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (1)$$

Power series have variable terms (functions of $z$), but *if we fix $z$, then all the concepts for series with constant terms in the last section apply*.





**THEOREM 1**

**Convergence of a Power Series**

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \quad (1)$$

**(a)** *Every power series* (1) *converges at the center* $z_0$.

**(b)** *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* $z$ *closer to* $z_0$ *than* $z_1$, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 365.

**(c)** *If* (1) *diverges at* $z = z_2$, *it diverges for every* $z$ *farther away from* $z_0$ *than* $z_2$. See Fig. 365.

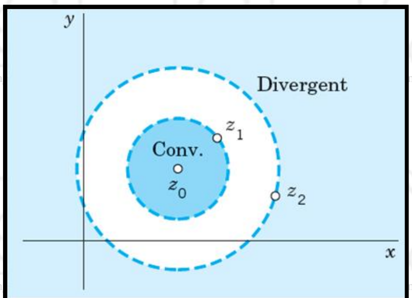


**Fig. 365.** Theroem 1

## Complex Analysis

### Radius of Convergence of a Power Series

consider the ***smallest*** circle with center $z_0$ that includes all the points at which a given power series (1) converges. Let $R$ denote its radius. The circle

$$|z - z_0| = R \qquad \text{(Fig. 366)}$$

is called the **circle of convergence** and its radius $R$ the **radius of convergence** of (1).

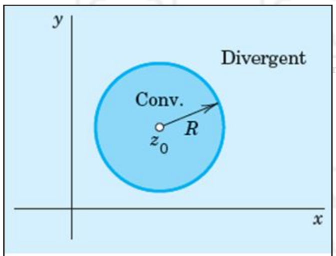


**Fig. 366.** Circle of convergence



## Complex Analysis

### THEOREM 2

**Radius of Convergence *R***

*Suppose that the sequence* $|a_{n+1}/a_n|$, $n = 1, 2, \cdots$, *converges with limit* $L^*$. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* $z$. *If* $L^* \neq 0$ (*hence* $L^* > 0$), *then*

$$\textbf{(6)} \qquad R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad \textbf{(Cauchy–Hadamard formula}^1\textbf{)}.$$

*If* $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ (*convergence only at the center* $z_0$).

# Functions Given by Power Series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

**THEOREM 1**

**Continuity of the Sum of a Power Series**

*If a function $f(z)$ can be represented by a power series (2) with radius of convergence $R > 0$, then $f(z)$ is continuous at $z = 0$.*

**THEOREM 2**

**Identity Theorem for Power Series. Uniqueness**

*Let the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ and $b_0 + b_1 z + b_2 z^2 + \cdots$ both be convergent for $|z| < R$, where $R$ is positive, and let them both have the same sum for all these $z$. Then the series are identical, that is, $a_0 = b_0, a_1 = b_1, a_2 = b_2, \cdots$.*

*Hence if a function $f(z)$ can be represented by a power series with any center $z_0$, this representation is **unique**.*





**THEOREM 3**

**Termwise Differentiation of a Power Series**

*The derived series of a power series has the same radius of convergence as the original series.*

We call **derived series** *of the power series* (2) the power series obtained from (2) by termwise differentiation, that is,

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots. \tag{3}$$

## Complex Analysis

**THEOREM 4**

**Termwise Integration of Power Series**

*The power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

*obtained by integrating the series* $a_0 + a_1 z + a_2 z^2 + \cdots$ *term by term has the same radius of convergence as the original series.*

### Power Series Represent Analytic Functions

**THEOREM 5**

**Analytic Functions. Their Derivatives**

*A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.*



## Complex Analysis

Summary:

-We can differentiate and integrate power series term by term (Theorems 3 & 4).

- Theorem 5: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus in turn are analytic functions.

In the next section we show that, conversely, *every* given analytic function can be represented by power series, called *Taylor series* and being the complex analog of the real Taylor series of calculus.

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# Taylor and Maclaurin Series





The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

**(1)** $$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or,

**(2)** $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*.$$

In (2) we integrate counterclockwise around a simple closed path $C$ that contains $z_0$ in its interior and is such that $f(z)$ is analytic in a domain containing $C$ and every point inside $C$.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

## Complex Analysis

The **remainder** of the Taylor series (1) after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^* \quad (3)$$

**THEOREM 1**

**Taylor's Theorem**

*Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor series (1) with center $z_0$ that represents $f(z)$. This representation is valid in the largest open disk with center $z_0$ in which $f(z)$ is analytic. The remainders $R_n(z)$ of (1) can be represented in the form (3). The coefficients satisfy the inequality*

$$|a_n| \leqq \frac{M}{r^n} \quad (5)$$

*where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.*

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**Accuracy of Approximation.** We can achieve any preassinged accuracy in approximating $f(z)$ by a partial sum of (1) by choosing $n$ large enough.

$$\lim_{n\to\infty} R_n(z) = 0.$$

## Important Special Taylor Series

**EXAMPLE 1** **Geometric Series**

Let $f(z) = 1/(1 - z)$.

$f^{(n)}(z) = n!/(1 - z)^{n+1}$, → $f^{(n)}(0) = n!$.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad (|z| < 1). \quad (11)$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence.

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**EXAMPLE 2** **Exponential Function**

$f(z) = e^z$ analytic for all $z$,

$(e^z)' = e^z$.

we obtain the Maclaurin series

(12) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots.$$



Complex Analysis

**EXAMPLE 3** **Trigonometric and Hyperbolic Functions**

(14)
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots.$$

(15)
$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$
$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots.$$

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**EXAMPLE 4 Logarithm**

From (1) it follows that

$$\text{Ln}\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \quad (|z| < 1). \tag{16}$$

Replacing $z$ by $-z$ and multiplying both sides by $-1$, we get

$$-\text{Ln}\,(1-z) = \text{Ln}\,\frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \quad (|z| < 1). \tag{17}$$

By adding both series we obtain

$$\text{Ln}\,\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \quad (|z| < 1). \tag{18}$$





## Practical Methods

**EXAMPLE 5 Substitution**

Find the Maclaurin series of $f(z) = 1/(1+z^2)$.

***Solution.*** By substituting $-z^2$ for $z$ in (11) we obtain

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \quad (|z| < 1). \tag{19}$$

**EXAMPLE 6 Integration** Find the Maclaurin series of $f(z) = \arctan z$.

***Solution.*** $f'(z) = 1/(1+z^2)$.

Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \quad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$

# SUMMARY OF CHAPTER

## Power Series, Taylor Series





A **power series** is of the form

$$(1) \qquad \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots;$$

**radius of convergence.** $|z - z_0| < R$

*every* analytic function $f(z)$ can be represented by power series.

**Taylor series** of $f(z)$ are of the form

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \qquad (|z-z_0)| < R),$$

# Laurent Series. Residue Integration





## Main Goal:

The main purpose of this chapter is to learn about another powerful method for evaluating complex integrals and certain real integrals.

It is called ***residue integration.***

*The main tool is the generalized Taylor Series, i.e. "Laurent Series".*

Laurent series generalize Taylor series. If, in an application, we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at $z_0$ (as defined in Sec. 15.4), we cannot use a Taylor series. Instead we can use a new kind of series, called **Laurent series**,[1] consisting of positive integer powers of $z - z_0$ (and a constant) as well as ***negative integer powers*** of $z - z_0$; this is the new feature.

## Complex Analysis

**THEOREM 1**

**Laurent's Theorem**

*Let $f(z)$ be analytic in a domain containing two concentric circles $C_1$ and $C_2$ with center $z_0$ and the annulus between them* (blue in Fig. 370). *Then $f(z)$ can be represented by the Laurent series*

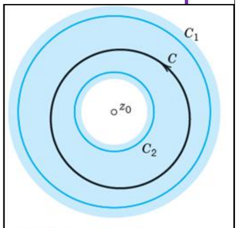


**Fig. 370.** Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{1}$$
$$\cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

*consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals*

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^*, \quad b_n = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*)\,dz^*, \tag{2}$$

*taken counterclockwise around any simple closed path $C$ that lies in the annulus and encircles the inner circle,* as in Fig. 370. [The variable of integration is denoted by $z^*$ since $z$ is used in (1).]



## Complex Analysis

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{1}$$
$$\cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

*This series converges and represents $f(z)$ in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle $C_1$ and decreasing $C_2$ until each of the two circles reaches a point where $f(z)$ is singular.*

*In the important special case that $z_0$ is the only singular point of $f(z)$ inside $C_2$, this circle can be shrunk to the point $z_0$, giving convergence in a disk except at the center. In this case the series (or finite sum) of the negative powers of* (1) *is called the* **principal part** *of $f(z)$ at $z_0$* [*or of that Laurent series* (1)].

## Complex Analysis

**COMMENT.** Obviously, instead of (1), (2) we may write (denoting $b_n$ by $a_{-n}$)

(1′)
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where all the coefficients are now given by a single integral formula, namely,

(2′)
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

**Uniqueness.**
*The Laurent series of a given analytic function f(z) in its annulus of convergence is unique. However, f(z) may have different Laurent series in two annuli with the same center;*



## Complex Analysis

**EXAMPLE 1** **Use of Maclaurin Series**

Find the Laurent series of $z^{-5} \sin z$ with center 0.

***Solution.*** By (14),

$$z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2 + - \cdots \qquad (|z| > 0).$$

Here the "annulus" of convergence is the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6} z^{-2}$.

## Complex Analysis

**EXAMPLE 2 Substitution**

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

***Solution.*** From (12)

with $z$ replaced by $1/z$ we obtain

$$z^2 e^{1/z} = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \qquad (|z| > 0).$$

a Laurent series whose principal part is an infinite series.

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## Complex Analysis

**EXAMPLE 3 Development of $1/(1 - z)$**

Develop $1/(1 - z)$ **(a)** in nonnegative powers of $z$, **(b)** in negative powers of $z$.

***Solution.***

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (\text{valid if } |z| < 1).$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad (\text{valid if } |z| > 1).$$

**EXAMPLE 4 Laurent Expansions in Different Concentric Annuli**

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

***Solution.*** Multiplying by $1/z^3$, we get from Example 3

(I) $$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1),$$

(II) $$\frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$

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## Complex Analysis

**EXAMPLE 5 Use of Partial Fractions**

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z+3}{z^2-3z+2}$ with center 0.

***Solution.***

In terms of partial fractions, $f(z) = -\dfrac{1}{z-1} - \dfrac{1}{z-2}$.

(a) and (b) in Example 3 take care of the first fraction.

For the second fraction,

(c) $$-\frac{1}{z-2} = \frac{1}{2\left(1-\frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

(d) $$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$



## Complex Analysis

(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 371),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots.$$

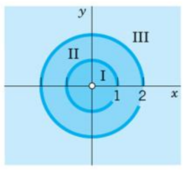


**Fig. 371.** Regions of convergence in Example 5

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots.$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1)\frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$

If $f(z)$ in Laurent's theorem is analytic inside $C_2$, the coefficients $b_n$ in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

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Questions? **Discussion?** Suggestions ?



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