In the name of God

دانشگاه زنجان

Engineering Mathematics

(Lecture # 19)

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Complex Analysis

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Laurent Series.
Residue Integration

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Main Goal:

The main purpose of this chapter is to learn about another powerful method for evaluating complex integrals and certain real integrals.

It is called *residue integration.*

The main tool is the generalized Taylor Series, i.e. "Laurent Series".

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COMMENT. Obviously, instead of (1), (2) we may write (denoting b_n by a_{-n})

$$(1') \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where all the coefficients are now given by a single integral formula, namely,

$$(2') \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Uniqueness.

The Laurent series of a given analytic function f(z) in its annulus of convergence is unique. However, f(z) may have different Laurent series in two annuli with the same center;

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EXAMPLE 3 **Development of $1/(1-z)$**

Develop $1/(1-z)$ **(a)** in nonnegative powers of z, **(b)** in negative powers of z.

Solution.

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1\text{)}.$$

(b) $$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1\text{)}.$$

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EXAMPLE 5 **Use of Partial Fractions**

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z+3}{z^2-3z+2}$ with center 0.

Solution.

In terms of partial fractions, $$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}.$$

(a) and (b) in Example 3 take care of the first fraction.

For the second fraction,

(c) $$-\frac{1}{z-2} = \frac{1}{2\left(1-\frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

(d) $$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

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(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 371),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots.$$

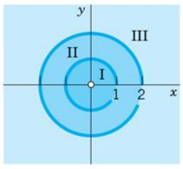

Fig. 371. Regions of convergence in Example 5

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots.$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

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Singularities and Zeros.

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* $\tan z$ has isolated singularities at $\pm\pi/2$, $\pm 3\pi/2$, etc.;

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Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

If the "principal part" has only finitely many terms, it is of the form:

$$(2) \qquad \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0).$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole**, and m is called its **order**.

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity**.

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EXAMPLE 1 **Poles. Essential Singularities**

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$.

THEOREM 1

Poles

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

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EXAMPLE 3 **Behavior Near an Essential Singularity**

$$\boxed{f(z) = e^{1/z}}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

has an essential singularity at $z = 0$.

It has **no limit** for approach along the imaginary axis;

it becomes infinite if $z \to 0$ through positive real values,

but it approaches zero if $z \to 0$ through negative real values.

It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small ϵ-neighborhood of $z = 0$.

set $z = re^{i\theta}$, $\Rightarrow$ $e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}$.

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$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}.$$

From
$$\left\{\begin{array}{l} \cos\theta = r\ln c_0, \quad -\sin\theta = \alpha r \\ \cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1 \end{array}\right\}$$
we obtain

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \quad \text{and} \quad \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Hence r can be made arbitrarily small by adding multiples of 2π to α,

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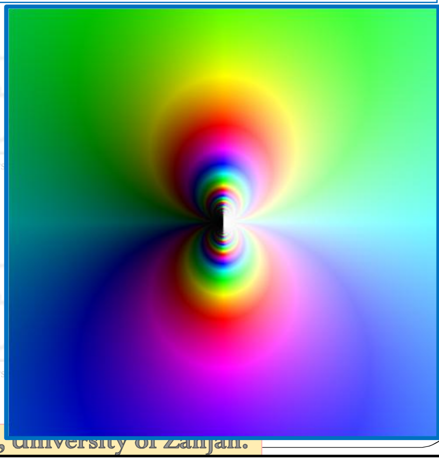
THEOREM 2

Picard's Theorem

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ϵ-neighborhood of z_0.

Plot of the function $f(z)=\exp(1/z)$, centered on the essential singularity at z=0. The hue of a point z represents the complex argument of $\exp(1/z)$, the luminance represents its absolute value. This plot shows that arbitrarily close to the singularity, all non-zero values are attained



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Zeros of Analytic Functions

A **zero** of an analytic function $f(z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$. A zero has **order** n if not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)}(z_0) \neq 0$. A first-order zero is also called a **simple zero**. For a second-order zero, $f(z_0) = f'(z_0) = 0$ but $f''(z_0) \neq 0$. And so on.

THEOREM 3

Zeros

The zeros of an analytic function $f(z)$ ($\not\equiv 0$) are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

THEOREM 4

Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of nth order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

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Riemann Sphere. Point at Infinity

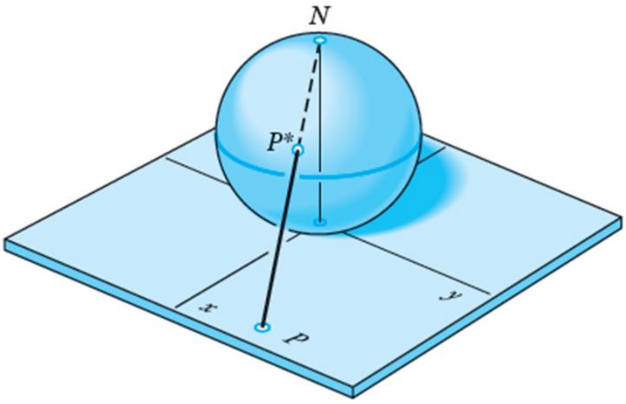

Fig. 372. Riemann sphere

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Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic** or **singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

(4) $$g(0) = \lim_{w \to 0} g(w)$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

EXAMPLE 5 **Functions Analytic or Singular at Infinity.**

$\boxed{f(z) = 1/z^2}$

is analytic at ∞ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$,

$f(z) = z^3$ is singular at ∞ and has a third-order pole there

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Questions? Discussion? Suggestions ?

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