In the name of God

Engineering Mathematics

(Lecture # 20)

Instructor:
Dr. Farhad Bayat

Zanjan University

Email: bayat.farhad@gmail.com

Complex Analysis

Laurent Series. Residue Integration

Lecturer: Dr Farhad Bayat, University of Zanjan.

Complex Analysis

Residue Integration Method

$$\oint_C f(z)\,dz$$ taken around a simple closed path C.

If $f(z)$ is analytic everywhere on C and inside C, → integral is zero

if $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Complex Analysis

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\,dz^*, \quad b_n = \frac{1}{2\pi i}\oint_C (z^* - z_0)^{n-1} f(z^*)\,dz^*,$$

Now comes the key idea.

The coefficient b_1 of this Laurent series is given by $b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz.$

since we can obtain Laurent series by various methods, without using the integral

→ **(1)** $$\oint_C f(z)\,dz = 2\pi i b_1.$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Complex Analysis

The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we denote it by

(2) $$b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

EXAMPLE 1 **Evaluation of an Integral by Means of a Residue**

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Solution.

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!} - \frac{z^3}{7!} + - \cdots \qquad |z| > 0$$

$f(z)$ has a pole of third order at $z = 0$

residue $b_1 = -\frac{1}{3}$ → $$\oint_C \frac{\sin z}{z^4}\,dz = 2\pi i b_1 = -\frac{\pi i}{3}.$$

Complex Analysis

EXAMPLE 2 **CAUTION! Use the Right Laurent Series!**

Integrate $f(z) = 1/(z^3 - z^4)$ clockwise around the circle C: $|z| = \frac{1}{2}$.

Solution.

$z^3 - z^4 = z^3(1 - z)$ → $f(z)$ is singular at $z = 0$ and $z = 1$.

$z = 1$ lies outside C.

So we need the residue of $f(z)$ at 0.

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad 0 < |z| < 1.$$

Clockwise integration thus yields → $$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

Formulas for Residues

Simple Poles at z_0.

(3) $$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z - z_0) f(z).$$

$f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0,

(4) $$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

EXAMPLE 3 **Residue at a Simple Pole**

$$f(z) = (9z + i)/(z^3 + z)$$

simple pole at i because $z^2 + 1 = (z + i)(z - i)$,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z\to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

Similarly: $p(i) = 9i + i$ and $q'(z) = 3z^2 + 1$ we confirm the result,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

Complex Analysis

Poles of Any Order at z_0. The residue of $f(z)$ at an mth-order pole at z_0 is

(5) $$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}.$$

In particular, for a second-order pole ($m = 2$),

(5*) $$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \{[(z-z_0)^2 f(z)]'\}.$$

Complex Analysis

EXAMPLE 4 Residue at a Pole of Higher Order

$$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$$

the denominator equals $(z+4)(z-1)^2$

second order at $z = 1$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1} \frac{d}{dz}\left(\frac{50z}{z+4}\right) = \frac{200}{5^2} = 8.$$

Complex Analysis

Several Singularities Inside the Contour.

THEOREM 1

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

Complex Analysis

PROOF

We enclose each of the singular points z_j in a circle C_j with radius small enough

Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$

From Cauchy's integral theorem

$$\oint_C f(z)\,dz + \oint_{-C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz + \cdots + \oint_{-C_k} f(z)\,dz = 0, \tag{7}$$

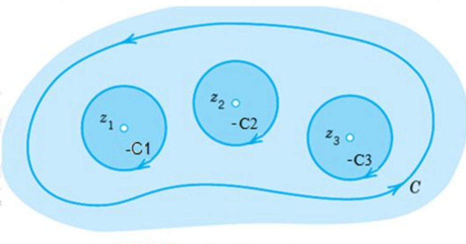

Fig. 373. Residue theorem

Lecturer: Dr Farhad Bayat, University of Zanjan.

Complex Analysis

(8) $$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz$$

where all the integrals are now taken counterclockwise.

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z),$$

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z).$$

Lecturer: Dr Farhad Bayat, University of Zanjan.

Complex Analysis

EXAMPLE 5 Integration by the Residue Theorem. Several Contours

(a) 0 and 1 are inside

(b) 0 is inside, 1 outside,

(c) 1 is inside, 0 outside,

(d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2-z}\,dz$$

Solution.

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z-1}\right]_{z=0} = -4,$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

(a) $2\pi i(-4+1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0.

Lecturer: Dr Farhad Bayat, University of Zanjan.

Questions? Discussion? Suggestions ?

Lecturer: Dr Farhad Bayat, University of Zanjan.