In the name of God

Engineering Mathematics

(Lecture # 21)

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Complex Analysis

Laurent Series. Residue Integration

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Complex Analysis

Residue Integration Method

$\oint_C f(z)\,dz$ taken around a simple closed path C.

If $f(z)$ is analytic everywhere on C and inside C, → integral is zero

Complex Analysis

Several Singularities Inside the Contour.

THEOREM 1

Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

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EXAMPLE 6

Integrate $(\tan z)/(z^2 - 1)$ counterclockwise around the circle C: $|z| = \frac{3}{2}$.

Solution.

$\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C.

$z^2 - 1 = (z-1)(z+1)$ → simple poles at ± 1.

$$\oint_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1}\frac{\tan z}{z^2-1} + \operatorname*{Res}_{z=-1}\frac{\tan z}{z^2-1}\right)$$

$$= 2\pi i\left(\frac{\tan z}{2z}\bigg|_{z=1} + \frac{\tan z}{2z}\bigg|_{z=-1}\right)$$

$$= 2\pi i \tan 1 = 9.7855i.$$

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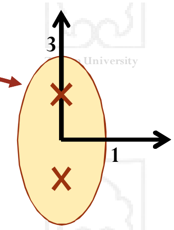
EXAMPLE 7 **Poles and Essential Singularities**

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z}\right) dz.$$



Solution.

$z^4 - 16 = 0$ at $\pm 2i$ and ± 2,

Inside

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

→

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16},$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4-16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

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The second term of the integrand has **an essential singularity at 0**, with residue:

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots$$

residue $\quad (|z| > 0).$

$$\oint_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z}\right) dz. = 2\pi i(-\tfrac{1}{16} - \tfrac{1}{16} + \tfrac{1}{2}\pi^2)$$

Complex Analysis

Residue Integration of Real Integrals

Integrals of Rational Functions of cos θ and sin θ

first consider integrals of the type

$$(1) \qquad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

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Setting $e^{i\theta} = z$, we obtain

(2)
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

$dz/d\theta = ie^{i\theta}$, → $d\theta = dz/iz$

(3)
$$J = \oint_C f(z)\,\frac{dz}{iz}$$

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θ ranges from 0 to 2π in (1), →

$z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$.

(3)
$$J = \oint_C f(z)\,\frac{dz}{iz}$$

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EXAMPLE 1 An Integral of the Type (1)

Show $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

Solution.

$\cos\theta = \frac{1}{2}(z + 1/z)$

$d\theta = dz/iz.$

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

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$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

$z_1 = \sqrt{2} + 1$ outside the unit circle C,

$z_2 = \sqrt{2} - 1$

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$

$$= -\frac{1}{2}.$$

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = 2\pi i(-2/i)(-\tfrac{1}{2}) = 2\pi.$$

Complex Analysis

As another large class, let us consider real integrals of the form

(4) **improper integral,** $$\int_{-\infty}^{\infty} f(x)\,dx.$$

(5′) $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)\,dx - \lim_{b\to\infty}\int_{0}^{b} f(x)\,dx.$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \quad \textbf{(5)}$$

We assume that the ***f(x)*** in (4) is a **real rational function** whose denominator is <u>different from zero for all real *x*</u> and is of degree **at least two units higher than** the degree of the numerator. Then the limits in exist, and we may start from (5).

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We consider the corresponding contour integral

(5*) $$\oint_C f(z)\,dz$$ C in Fig. 374.

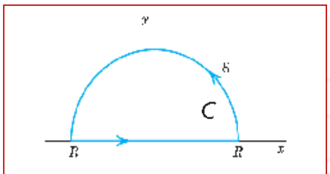

Fig. 374. Path C of the contour integral in (5*)

if we choose **R large enough**, then C encloses all poles.
By the residue theorem we then obtain:

$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

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(6) $$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) - \int_{S} f(z)\,dz.$$

We prove that, if $R \to \infty$, ➡ $\int_S f(z)\,dz \to 0$ $\quad z = Re^{i\theta}$

Since the degree of the denominator of $f(z)$ is at least **two units higher than** the degree of the numerator, we have:

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0.

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By the ML-inequality

$$\left|\int_S f(z)\,dz\right| < \frac{k}{R^2}\pi R = \frac{k\pi}{R} \qquad (R > R_0).$$

if $R \to \infty$, ➡ $\int_S f(z)\,dz \to 0$

(7) $$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z)$$

the poles of $f(z)$ in the upper half-plane.

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EXAMPLE 2 **An Improper Integral from 0 to ∞**

show that $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

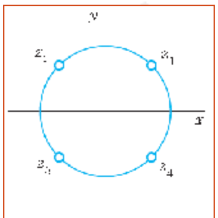

Fig. 375. Example 2

Solution.

$f(z) = 1/(1+z^4)$ → $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{-3\pi i/4}$, $z_4 = e^{-\pi i/4}$.

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}.$$

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$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = -\frac{2\pi i}{4}\left(e^{\pi i/4} - e^{-\pi i/4}\right) = -\frac{2\pi i}{4}\cdot 2i \sin\frac{\pi}{4} = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1+x^4)$ is an even function, →

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Questions? **Discussion?** Suggestions ?

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