In the name of God

Engineering Mathematics

(Lecture # 22)

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Residue Integration of Real Integrals

Complex Analysis

Residue Integration of Real Integrals

Integrals of Rational Functions of cos θ and sin θ

first consider integrals of the type

$$(1) \qquad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

Setting $e^{i\theta} = z$, we obtain

$$(3) \qquad J = \oint_C f(z) \frac{dz}{iz}$$

Complex Analysis

As another large class, let us consider real integrals of the form

$$(4) \qquad \textbf{improper integral,} \quad \int_{-\infty}^{\infty} f(x)\, dx.$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \qquad \textbf{(5)}$$

We assume that the $f(x)$ in (4) is a **real rational function** whose denominator is different from zero for all real x and is of degree **at least two units higher than** the degree of the numerator.

$$(7) \qquad \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z)$$

the poles of $f(z)$ in the upper half-plane.

Fourier Integrals

$$(8) \qquad \int_{-\infty}^{\infty} f(x) \cos sx \, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x) \sin sx \, dx$$

the contour C in Fig. 374.

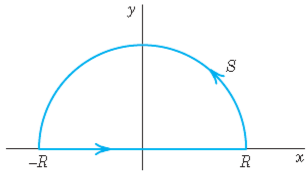

Fig. 374. Path C of the contour integral in (8)

Similar to (7):

$$(9) \qquad \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = 2\pi i \sum \operatorname{Res} [f(z) e^{isz}]$$

sum the residues of $f(z)e^{isz}$ at its poles in the upper half-plane.

Therefore we get:

$$(10) \qquad \begin{aligned} \int_{-\infty}^{\infty} f(x) \cos sx \, dx &= -2\pi \sum \operatorname{Im} \operatorname{Res} [f(z) e^{isz}], \\ \int_{-\infty}^{\infty} f(x) \sin sx \, dx &= 2\pi \sum \operatorname{Re} \operatorname{Res} [f(z) e^{isz}]. \end{aligned}$$

Complex Analysis

EXAMPLE 3 **An Application of (10)**

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k}e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0$$

$(s > 0, k > 0)$.

Solution.

only one pole in the upper half-plane, namely, a simple pole at $z = ik$,

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2+z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2+x^2}\,dx = 2\pi i\,\frac{e^{-ks}}{2ik} = \frac{\pi}{k}e^{-ks}.$$

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Complex Analysis

Another Kind of Improper Integral

$$(11) \qquad \int_A^B f(x)\,dx$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x\to a} |f(x)| = \infty.$$

By definition, this integral (11) means

$$(12) \qquad \int_A^B f(x)\,dx = \lim_{\epsilon\to 0}\int_A^{a-\epsilon} f(x)\,dx + \lim_{\eta\to 0}\int_{a+\eta}^{B} f(x)\,dx$$

where both ϵ and η approach zero independently and through positive values.

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It may happen that neither of two limits exists if ϵ and η go to 0 independently, but the following limit exists:

$$\lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\,dx + \int_{a+\epsilon}^{B} f(x)\,dx \right] \tag{13}$$

This is called the **Cauchy principal value** of the integral.

$$\text{pr. v.} \int_A^B f(x)\,dx.$$

For example,

$$\text{pr. v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3} \right] = 0;$$

the principal value exists, although the integral itself has no meaning.

Complex Analysis

In the case of ***simple poles on the real axis*** we shall obtain a formula for the *principal value of an integral from* -∞ *to* +∞.

THEOREM 1

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 376)

$$\lim_{r \to 0} \int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

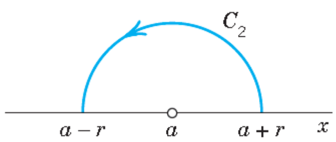


Fig. 376. Theorem 1

Complex Analysis

Then the desired formula is

$$\text{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \text{Res}\, f(z) + \pi i \sum \text{Res}\, f(z)$$

over all poles in the ***upper half-plane***

over all ***poles on the real axis***

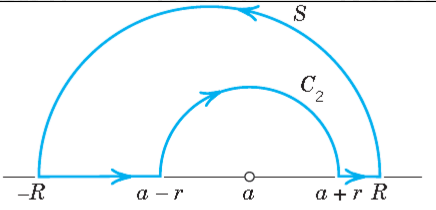


Fig. 377. Application of Theorem 1

Complex Analysis

EXAMPLE 4 **Poles on the Real Axis**

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution.

$$x^2 - 3x + 2 = (x-1)(x-2),$$

simple poles at

$$z = 1, \qquad \underset{z=1}{\text{Res}}\, f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \underset{z=2}{\text{Res}}\, f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5},$$

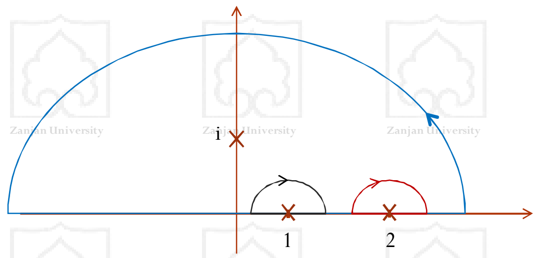
$$z = i, \qquad \underset{z=i}{\text{Res}}\, f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20},$$

$z = -i$ in the lower half-plane, which is of no interest here.

Complex Analysis

From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10}.$$



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Complex Analysis

SUMMARY OF CHAPTER

A **Laurent series** is a series of the form

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

or, more briefly written

$$(1^*) \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\,dz^*$$

The negative powers in this Laurent series is called the **principal part** of *f(z)* at z_0.

The coefficient b_1 of $1/(z - z_0)$ in this series is called the **residue**

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SUMMARY OF CHAPTER

$$(2)\quad b_1 = \operatorname*{Res}_{z\to z_0} f(z) = \frac{1}{2\pi i}\oint_C f(z^*)\,dz^*.$$

Thus $\displaystyle \oint_C f(z^*)\,dz^* = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$

$$(3)\quad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right),$$

provided $f(z)$ has at z_0 a **pole of order** $\boldsymbol{m}$;

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z);\qquad \operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

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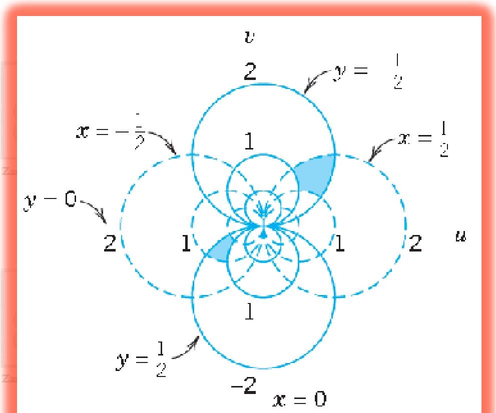
Conformal Mapping



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Conformal Mapping

Motivation:

Conformal mappings are invaluable to the **engineer** and **physicist** as an **aid in solving problems** in potential theory. They are a standard method for **solving** ***boundary value problems*** in two-dimensional potential theory and yield rich applications in **electrostatics**, **heat flow**, and **fluid flow**.

The main feature of conformal mappings is that they are **angle-preserving**.

if $f(z)$ is an **analytic** function, then the mapping given by $w=f(z)$ is a **conformal mapping**, that is, it **preserves angles**, except at points where the derivative $f'(z)$ is zero.

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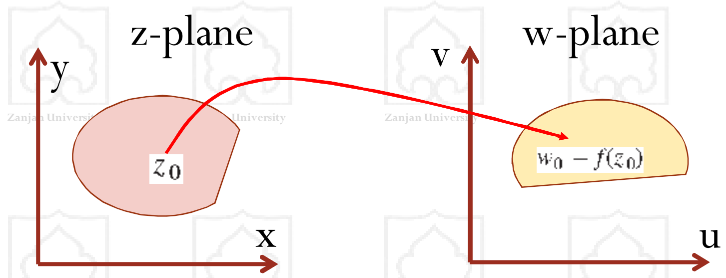
Conformal Mapping

Geometry of Analytic Functions: Conformal Mapping

A complex function

$$w = f(z) = u(x, y) - iv(x, y) \qquad (z = x - iy) \tag{1}$$



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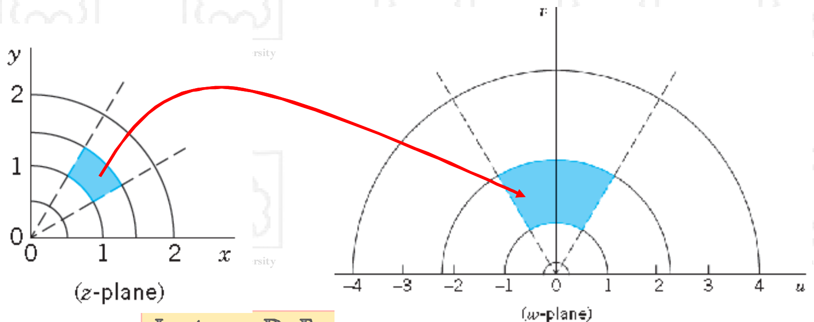
Conformal Mapping

EXAMPLE 1 Mapping $w = f(x) = z^2$

$z = re^{i\theta}$ and $w = Re^{i\phi}$, → $w = z^2 = r^2 e^{2i\theta}$.

$R = r^2$ $\phi = 2\theta$

Figure 378 shows this for the region $1 \leqq |z| \leqq \frac{3}{2}, \pi/6 \leqq \theta \leqq \pi/3$,



Conformal Mapping

$w = f(z) = u(x, y) + iv(x, y)$

In Cartesian coordinates we have $z = x + iy$ and

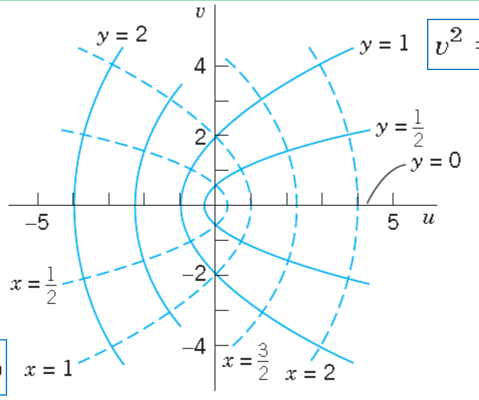
$u = \text{Re}\,(z^2) = x^2 - y^2$, $v = \text{Im}\,(z^2) = 2xy$.

vertical lines $x = c = \text{const}$ → $\begin{cases} u = c^2 - y^2 \\ v = 2cy. \end{cases}$

we can eliminate y. → $v^2 = 4c^2(c^2 - u)$

Similarly, horizontal lines $y = k = \text{const}$ → $v^2 = 4k^2(k^2 + u)$

Conformal Mapping



$v^2 = 4k^2(k^2 + u)$

$v^2 = 4c^2(c^2 - u)$

Fig. 379. Images of x = const, y = const under $w = z^2$

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Conformal Mapping

A mapping $w = f(z)$ is called **conformal** if it preserves angles between oriented curves in magnitude as well as in sense. Figure 380 shows what this means. The **angle** α $(0 \leqq \alpha \leqq \pi)$ between two intersecting curves C_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0. And *conformality* means that the images C_1^* and C_2^* of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.

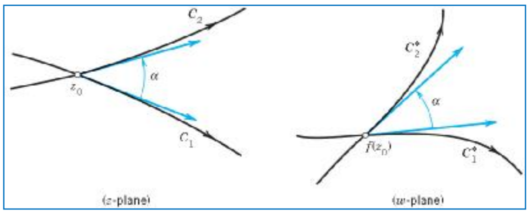

Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

THEOREM 1

Conformality of Mapping by Analytic Functions

The mapping $w = f(z)$ by an analytic function f is conformal, except at **critical points**, *that is, points at which the derivative f' is zero.*

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Conformal Mapping

EXAMPLE 2 Conformality of $w = z^n$

The mapping $w = z^n$, $n = 2, 3, \cdots$, is conformal, except at $z = 0$, where $w' = nz^{n-1} = 0$.

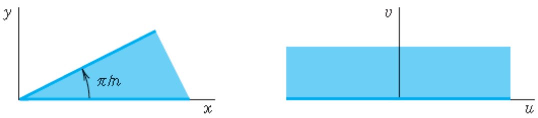

Fig. 382. Mapping by $w = z^n$

EXAMPLE 3 Mapping $w = z + 1/z$. Joukowski Airfoil

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta).$$

$$u = a\cos\theta, \qquad v = b\sin\theta$$

where $a = r + \frac{1}{r}$, $b = r - \frac{1}{r}$.

Conformal Mapping

$u = a\cos\theta, \qquad v = b\sin\theta$ where $a = r + \frac{1}{r}$, $b = r - \frac{1}{r}$.

circles $|z| = r = \text{const} \neq 1$ → $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$

circle $|z| = 1 = e^{i\theta}$ → segment $-2 \leqq u \leqq 2$

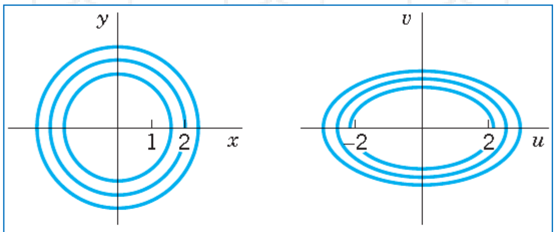

Fig. 383. Example 3

Conformal Mapping

Now the derivative of w is

$$w' = 1 - \frac{1}{z^2} = \frac{(z+1)(z-1)}{z^2}$$

which is 0 at $z = \pm 1$.

The larger circle is mapped onto a *Joukowski airfoil*. The dashed circle passes through both and 1 and -1 is mapped onto a curved segment.

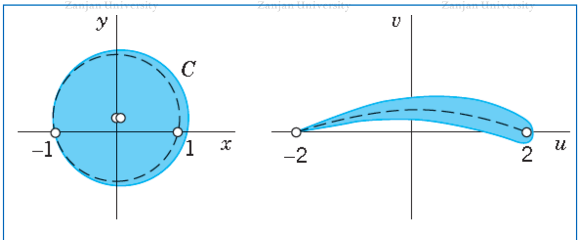

Fig. 384. Joukowski airfoil

Conformal Mapping

EXAMPLE 4 Conformality of $w = e^z$

we have $|e^z| = e^x$ and $\text{Arg } w = y$

line $x = x_0 = \text{const}$ → circle $|w| = e^{x_0}$

line $y = y_0 = \text{const}$ → $\arg w = y_0$.

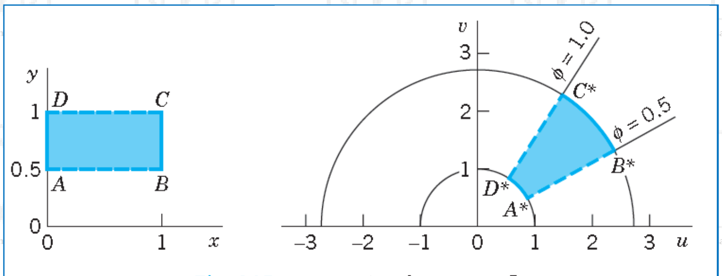

Fig. 385. Mapping by $w = e^z$

Conformal Mapping

The fundamental region $-\pi < \operatorname{Arg} z \leqq \pi$ of e^z

entire w-plane without the origin $w = 0$

(because $e^z = 0$ for no z).

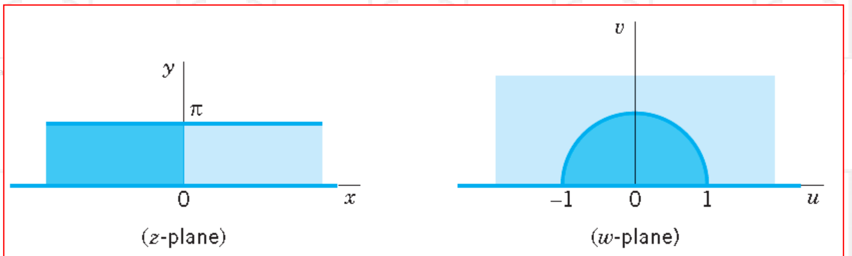

Fig. 386. Mapping by $w = e^z$

Questions? Discussion? Suggestions ?