In the name of God

Engineering Mathematics

(Lecture # 23)

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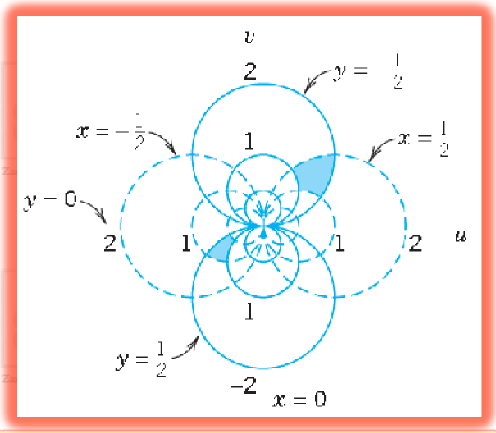
Conformal Mapping



Conformal Mapping

Motivation:

Conformal mappings are invaluable to the **engineer** and **physicist** as an **aid in solving problems** in potential theory. They are a standard method for **solving *boundary value problems*** in two-dimensional potential theory and yield rich applications in **electrostatics**, **heat flow**, and **fluid flow**.

The main feature of conformal mappings is that they are **angle-preserving**.

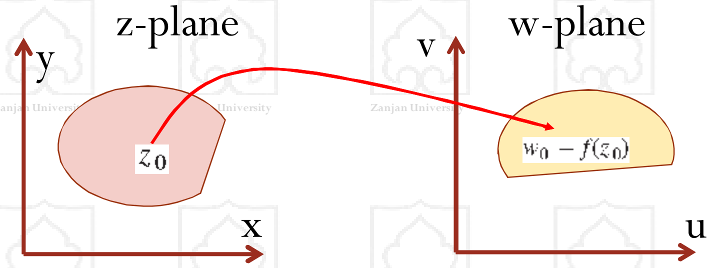
if $f(z)$ is an **analytic** function, then the mapping given by $w=f(z)$ is a **conformal mapping**, that is, it **preserves angles**, except at points where the derivative $f'(z)$ is zero.

Conformal Mapping

Geometry of Analytic Functions: Conformal Mapping

A complex function

(1) $$w = f(z) = u(x, y) - iv(x, y) \qquad (z = x - iy)$$



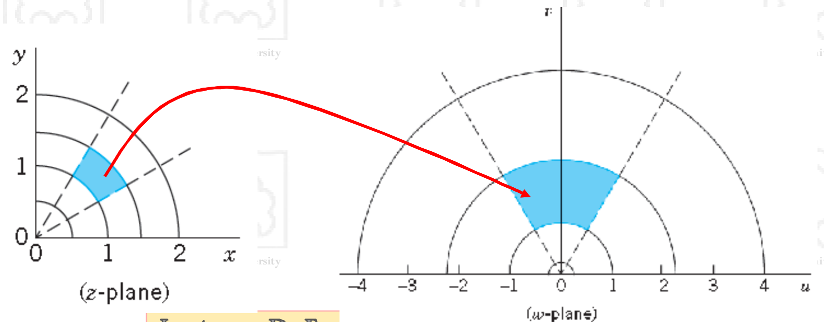
Conformal Mapping

EXAMPLE 1 **Mapping $w = f(z) = z^2$**

$z = re^{i\theta}$ and $w = Re^{i\phi}$, → $w = z^2 = r^2 e^{2i\theta}$.

$R = r^2$ $\quad$ $\phi = 2\theta$

Figure 378 shows this for the region $1 \leqq |z| \leqq \frac{3}{2}, \pi/6 \leqq \theta \leqq \pi/3$,



Conformal Mapping

$w = f(z) = u(x, y) + iv(x, y)$

In Cartesian coordinates we have $z = x + iy$ and

$u = \operatorname{Re}(z^2) = x^2 - y^2$, $\quad$ $v = \operatorname{Im}(z^2) = 2xy$.

vertical lines $x = c =$ const → $\begin{cases} u = c^2 - y^2 \\ v = 2cy. \end{cases}$

we can eliminate y. → $v^2 = 4c^2(c^2 - u)$

Similarly, horizontal lines $y = k =$ const → $v^2 = 4k^2(k^2 + u)$

Conformal Mapping

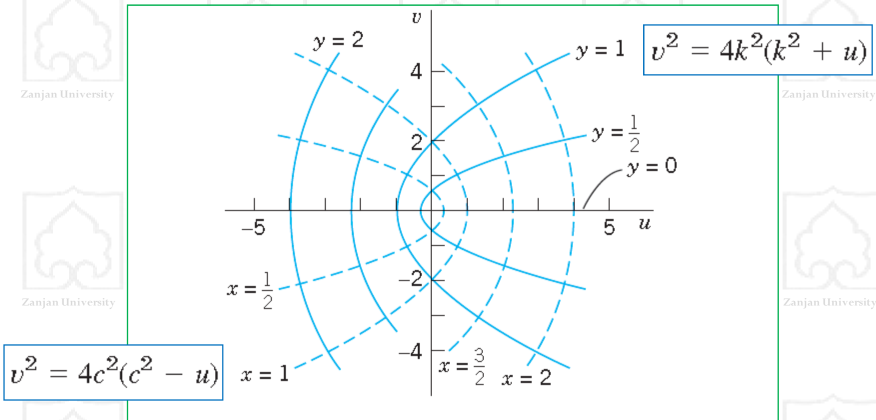


Fig. 379. Images of x = const, y = const under $w = z^2$

Conformal Mapping

A mapping $w = f(z)$ is called **conformal** if it preserves angles between oriented curves in magnitude as well as in sense. Figure 380 shows what this means. The **angle** α $(0 \leqq \alpha \leqq \pi)$ between two intersecting curves C_1 and C_2 is defined to be the angle between their oriented tangents at the intersection point z_0. And *conformality* means that the images C_1^* and C_2^* of C_1 and C_2 make the same angle as the curves themselves in both magnitude and direction.

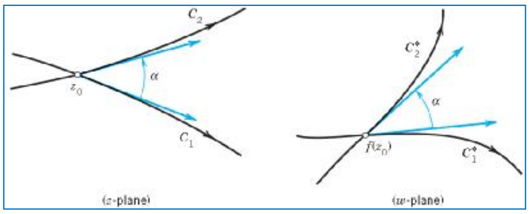


Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

THEOREM 1

Conformality of Mapping by Analytic Functions

The mapping $w = f(z)$ by an analytic function f is conformal, except at **critical points**, *that is, points at which the derivative f' is zero.*

Conformal Mapping

EXAMPLE 2 Conformality of $w = z^n$

The mapping $w = z^n, n = 2, 3, \cdots$, is conformal, except at $z = 0$, where $w' = nz^{n-1} = 0$.

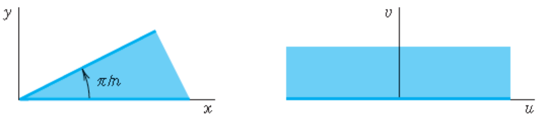


Fig. 382. Mapping by $w = z^n$

EXAMPLE 3 Mapping $w = z + 1/z$. Joukowski Airfoil

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta).$$

$$u = a\cos\theta, \qquad v = b\sin\theta$$

where $a = r + \frac{1}{r}, \quad b = r - \frac{1}{r}.$

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$$u = a\cos\theta, \qquad v = b\sin\theta$$ where $a = r + \frac{1}{r}, \quad b = r - \frac{1}{r}.$

circles $|z| = r = \text{const} \neq 1$ → $$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

circle $|z| = 1 = e^{i\theta}$ → segment $-2 \leqq u \leqq 2$

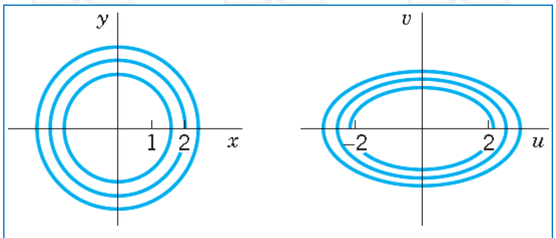

Fig. 383. Example 3

Conformal Mapping

The larger circle is mapped onto a ***Joukowski airfoil***. The dashed circle passes through both and 1 and -1 is mapped onto a curved segment.

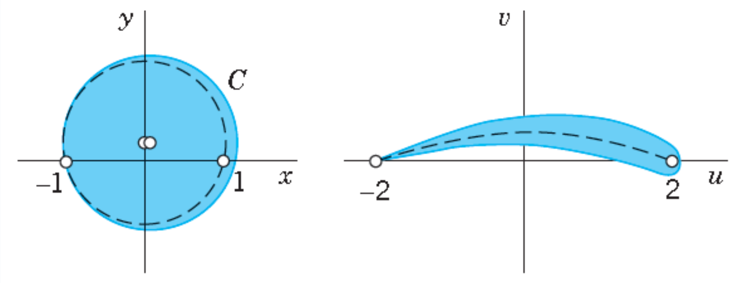

Fig. 384. Joukowski airfoil

Conformal Mapping

EXAMPLE 4 **Conformality of $w = e^z$**

we have $|e^z| = e^x$ and $\text{Arg } w = y$

line $x = x_0 = \text{const}$ → circle $|w| = e^{x_0}$

line $y = y_0 = \text{const}$ → $\arg w = y_0$.

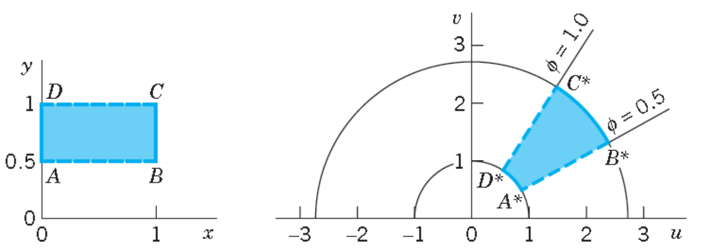

Fig. 385. Mapping by $w = e^z$

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The fundamental region $-\pi < \text{Arg}\, z \leqq \pi$ of e^z

entire w-plane without the origin $w = 0$

(because $e^z = 0$ for no z).

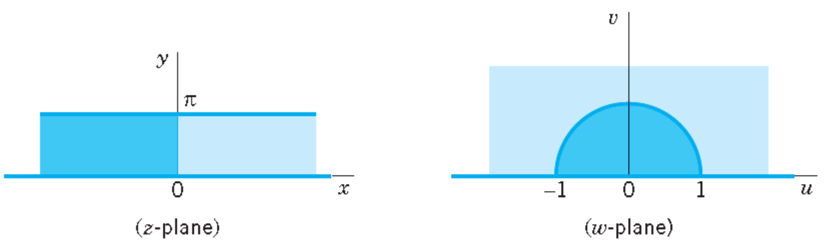

Fig. 386. Mapping by $w = e^z$

Conformal Mapping

Linear Fractional Transformations (Möbius Transformations)

Conformal Mapping

Linear fractional transformations (or **Möbius transformations**) are mappings

(1) $$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0)$$

where a, b, c, d are complex or real numbers. Differentiation gives

(2) $$w' = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2}.$$

requirement $ad - bc \neq 0$.

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Special cases of (1) are

(3)
$$\begin{aligned} w &= z + b && \text{(Translations)} \\ w &= az \quad \text{with } |a| = 1 && \text{(Rotations)} \\ w &= az + b && \text{(Linear transformations)} \\ w &= 1/z && \text{(Inversion in the unit circle).} \end{aligned}$$

$$w = \frac{az + b}{cz + d}$$

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EXAMPLE 1 Properties of the Inversion $w = 1/z$ (Fig. 387)

$z = re^{i\theta}$ and $w = Re^{i\phi}$

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \quad \Rightarrow \quad R = \frac{1}{r}, \quad \phi = -\theta.$$

the unit circle $|z| = r = 1$ is mapped onto the unit circle $|w| = R = 1$; $w = e^{i\phi} = e^{-i\theta}$.

$w = 1/z$ maps every straight line or circle onto a circle or straight line.

Conformal Mapping

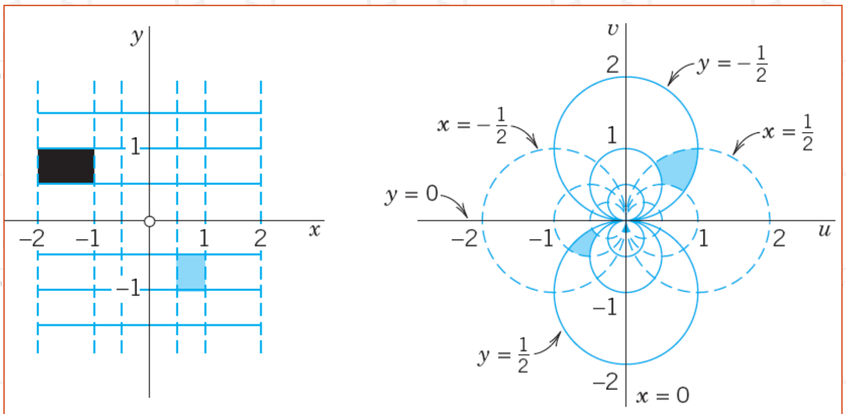

Fig. 387. Mapping (Inversion) $w = 1/z$

Conformal Mapping

Proof. Every straight line or circle in the z-plane can be written

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

$A = 0$ gives a straight line and $A \neq 0$ a circle.

$$Az\bar{z} + B\frac{z + \bar{z}}{2} + C\frac{z - \bar{z}}{2i} + D = 0.$$

Substitution of $z = 1/w$

$$A + B\frac{\bar{w} + w}{2} + C\frac{\bar{w} - w}{2i} + Dw\bar{w} = 0$$

$$A + Bu - Cv + D(u^2 + v^2) = 0.$$

a circle (if $D \neq 0$) or a straight line (if $D = 0$) in the w-plane.

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Conformal Mapping

THEOREM 1

Three Points and Their Images Given

Three given distinct points z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one, and only one, linear fractional transformation $w = f(z)$. This mapping is given implicitly by the equation

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}. \quad \textbf{(2)}$$

(If one of these points is the point ∞, the quotient of the two differences containing this point must be replaced by 1.)

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EXAMPLE 1 Mapping of a Half-Plane onto a Disk (Fig. 388)

Find the linear fractional transformation (1) that maps $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$, respectively.

Solution.

$$\frac{w-(-1)}{w-1}\cdot\frac{-i-1}{-i-(-1)} = \frac{z-(-1)}{z-1}\cdot\frac{0-1}{0-(-1)},$$

$$w = \frac{z-i}{-iz+1}.$$

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$z = x$ → $w = (x-i)/(-ix+1)$, thus $|w| = 1$, unit circle.

$z = i$ → $w = 0$,

$$w = \frac{z-i}{-iz+1}.$$

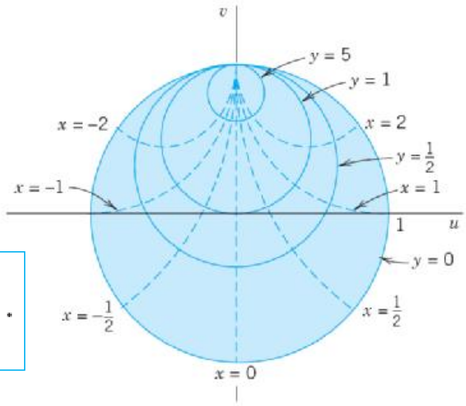

Fig. 388. Linear fractional transformation in Example 1

Conformal Mapping

EXAMPLE 2 **Occurrence of ∞**

Determine the linear fractional transformation that maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$, respectively.

Solution. From (2) we obtain the desired mapping

$$w = \frac{z - i}{z + i}.$$

(2)
$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

Conformal Mapping

EXAMPLE 3 **Mapping of a Disk onto a Half-Plane**

Find the linear fractional transformation that maps $z_1 = -1$, $z_2 = i$, $z_3 = 1$ onto $w_1 = 0$, $w_2 = i$, $w_3 = \infty$, respectively, such that the unit disk is mapped onto the right half-plane. (Sketch disk and half-plane.)

Solution.

From (2) we obtain, after replacing $(i - \infty)/(w - \infty)$ by 1,

$$w = -\frac{z + 1}{z - 1}.$$

(2)
$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

Conformal Mapping

Mappings of disks onto disks

the unit disk in the z-plane is mapped onto the unit disk in the w-plane by

(3) $$w = \frac{z - z_0}{cz - 1}, \qquad c = \bar{z}_0, \qquad |z_0| < 1.$$

Conformal Mapping

EXAMPLE 5 **Mapping of the Unit Disk onto the Unit Disk**

Taking $z_0 = \frac{1}{2}$ in (3), we obtain (verify!)

(3) $$w = \frac{z - z_0}{cz - 1}, \qquad c = \bar{z}_0,$$ → $$w = \frac{2z - 1}{z - 2}$$

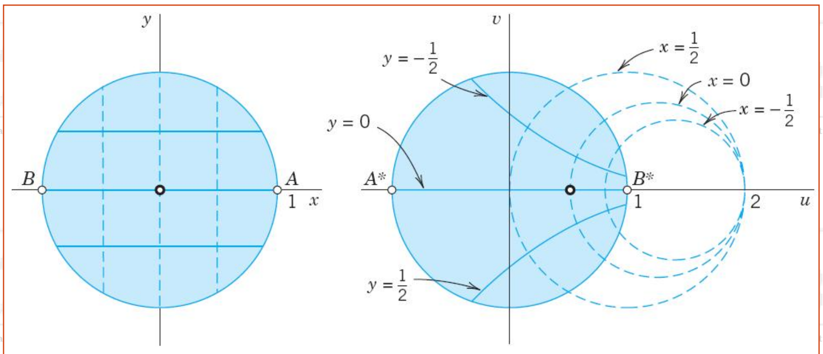

Fig. 389. Mapping in Example 5

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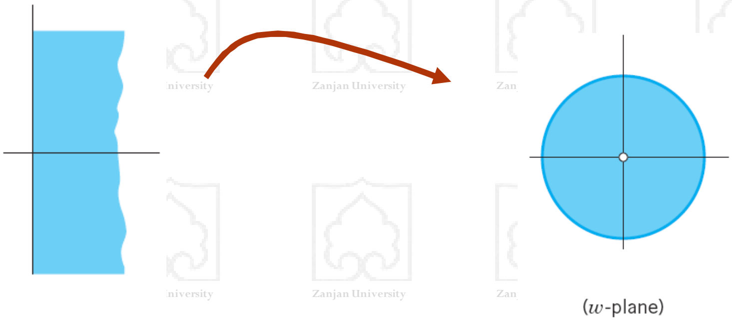
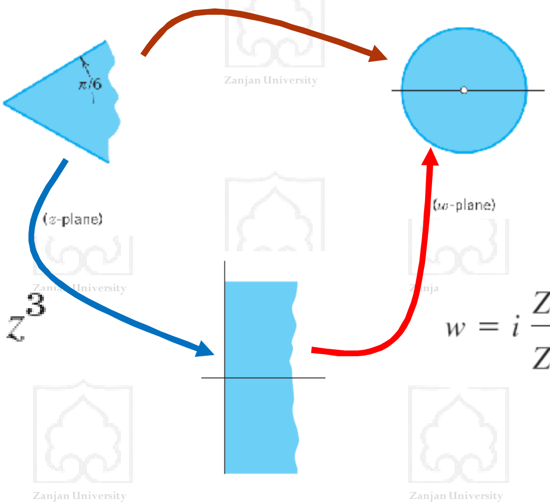
EXAMPLE

$$w = i\,\frac{z-1}{z+1}$$



Conformal Mapping

EXAMPLE 6 **Mapping of an Angular Region onto the Unit Disk**



$$Z = z^3$$

$$w = i\,\frac{Z-1}{Z+1},$$

Questions? Discussion? Suggestions ?

Conformal Mapping

Sine Function.

$$w = u + iv = \sin z = \sin x \cosh y + i \cos x \sinh y \quad (1)$$

Hence

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y. \quad (2)$$

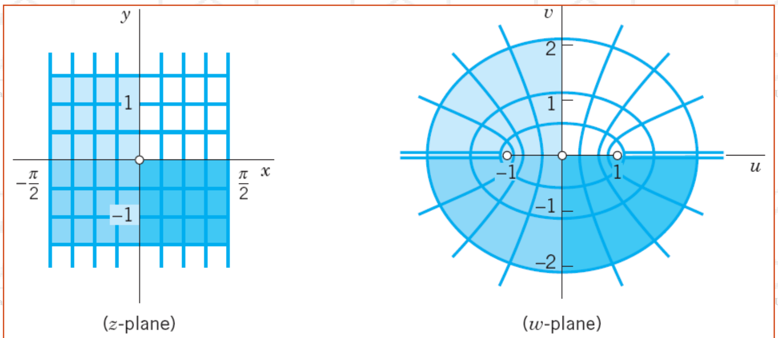


Fig. 391. Mapping $w = u + iv = \sin z$

Conformal Mapping

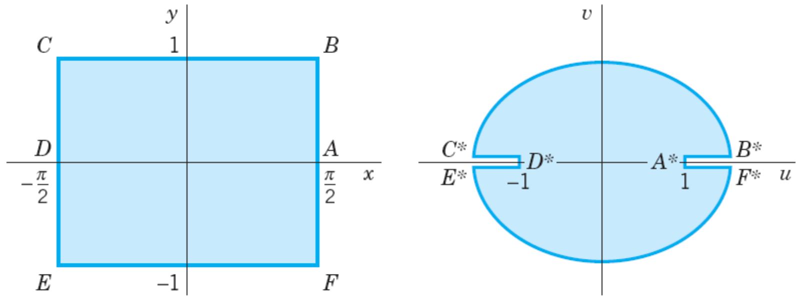


Fig. 392. Mapping by $w = \sin z$

Since $f'(z) = \cos z = 0$ at $z = \pm\frac{1}{2}\pi$, the mapping is not conformal at these two critical points.