# Mechanical Energy Change in Inertial Reference Frames

**Saeed Ghanbari,** University of Zanjan, Zanjan, Iran

The mechanical energy change of a system in an inertial frame of reference equals work done by the total nonconservative force in the same frame. This relation is covariant under the Galilean transformations from inertial frame $S$ to $S'$, where $S'$ moves with constant velocity relative to $S$. In the presence of nonconservative forces, such as normal and tension forces, the mechanical energy of a system can be conserved in $S$ and not be conserved in $S'$. In this paper we find useful relations between the mechanical energy changes in two inertial frames of reference.

## Introduction

The fact that the work done by an arbitrary force could have different values in different inertial frames of reference is not usually covered in the introductory physics textbooks and is neither well discussed nor explained in the physics classrooms. Furthermore, many high school and undergraduate students assume that, in the absence of the nonconservative force of friction, the mechanical energy of the system is conserved in all inertial frames of reference. Reading this paper could prevent some mistakes in common problems involving mechanical energy. The context for this study is the introductory physics course. The primary audience is prospective and practicing physics teachers. The contents of this paper, including the examples and exercises provided for the readers, could be used by physics teachers to teach the mechanical energy topic, to improve students' conceptual understanding of the subject, and to examine their students' learning.

The laws of Newtonian mechanics such as Newton's laws of motion are covariant under the Galilean transformations.[1] Also, the work-kinetic energy theorem, which is derived from Newton's second law, is covariant under these transformations.[2,3] It has been shown that, as long as the work of fictitious forces is properly included in the formalism, the work-kinetic energy theorem can be applied to an isolated system of particles in a rotating reference frame.[4] In an inertial frame of reference, when the work done by nonconservative forces is zero, we can use the principle of conservation of mechanical energy $E$, which is defined as the sum of kinetic energy $K$ and potential energy $U$.[5,6] Moreover, after investigating some examples, it has been argued that for isolated systems, the conservation of mechanical energy should not be dependent on any choice of any particular inertial frame of reference.[7] Contrary to those conclusions, we show that in the presence of nonconservative forces, conservation of mechanical energy can be violated under the Galilean transformations. In Newtonian mechanics, force is invariant, but the work done by a force depends on the inertial frame of reference. In an inertial frame of reference, in the presence of nonconservative forces such as frictional, normal, and tension forces, if the work done by nonconservative forces on a system is zero, the mechanical energy is conserved. The work done by the same nonconservative forces in another inertial reference frame may be nonzero. Therefore, in the second frame of reference mechanical energy is not conserved. Considering three examples, we discuss the results. In this work, problems involving rotational kinetic and potential energy changes are not studied.

## Covariance of $\Delta E = W_{nc}$

According to the work-kinetic energy theorem,[5] which is derived from Newton's second law, the kinetic energy change $\Delta K$ in $S$ frame equals work $W$ done by total force $\mathbf{F}$,

$$\Delta K = W, \tag{1}$$

where $W = W_c + W_{nc}$. The work $W_c$ done by conservative force $\mathbf{F}_c$ and the work $W_{nc}$ done by nonconservative force $\mathbf{F}_{nc}$ are given by $W_c = \int \mathbf{F}_c \cdot d\mathbf{r}$ and $W_{nc} = \int \mathbf{F}_{nc} \cdot d\mathbf{r}$, respectively. Therefore, considering Eq. (1) as well as definitions for the potential energy change $\Delta U = -W_c$ and mechanical energy $E = K+U$, we obtain[5]

$$\Delta E = W_{nc}. \tag{2}$$

Starting from Newton's second law in the inertial frame of reference $S'$, we find that $\Delta K' = W'$ and $\Delta E' = W'_{nc}$, which implies the covariance of Eqs. (1)[2,3] and (2) under the Galilean transformations. Thus, according to Eq. (2), in all inertial frames of reference, change of mechanical energy equals work done by nonconservative forces. This statement, which is as important as the work-kinetic energy theorem, can be used to directly calculate the mechanical energy change in any inertial frame of reference and may be called the "nonconservative work-mechanical energy theorem." According to this theorem, in a certain inertial frame of reference, mechanical energy is conserved if and only if $W_{nc} = 0$. Taking into account that $W_{nc}$ may have different values in different inertial frames, we come up with the result that the mechanical energy of a system can be conserved in an inertial frame of reference and not be conserved in another inertial reference frame.

## Relations between $\Delta E$ and $\Delta E'$

As shown in Fig. 1, according to the Galilean transformations between the inertial frames of reference $S$ and $S'$, we have $\mathbf{r}' = \mathbf{r} - \mathbf{R}_0 - \mathbf{v}_{0'}\, t$, where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of a particle of mass $m$, relative to $S$ and $S'$, respectively, and

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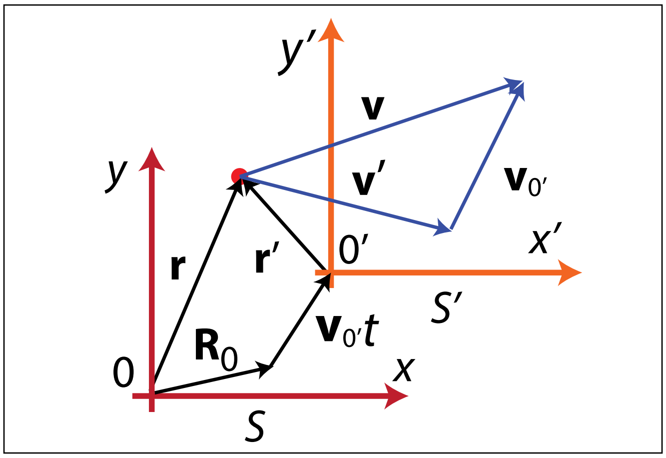


**Fig. 1. Galilean coordinates and velocity transformations are shown in this figure. Position vector r of a particle with respect to the inertial frame of reference *S* is related to the position vector of the particle relative to the inertial frame of reference S′ moving at a constant velocity $\mathbf{v}_{0'}$ with respect to the *S* frame. Here, the *z*-axis has been omitted for simplicity.**

$\mathbf{v}_{0'}$ is the velocity of $S'$ relative to $S$. Also, $\mathbf{R}_0$ is the initial position vector of the origin of $S'$ with respect to $S$. So, we obtain $d\mathbf{r}' = d\mathbf{r} - \mathbf{v}_{0'}\, dt$, which dividing by $dt$ leads to $\mathbf{v}' = \mathbf{v} - \mathbf{v}_{0'}$. Using a unit vector $\hat{u}$, we write $\mathbf{v}_{0'} = \hat{u} v_{0'}$ and obtain $v'^2 = v^2 + v_{0'}^2 - 2v_u v_{0'}$, where $v_u$ is the component of $\mathbf{v}$ along the unit vector $\hat{u}$, which is in the direction of the velocity of $S'$ with respect to $S$. We can write

$$\Delta K = \frac{1}{2}mv_{\mathrm{f}}^2 - \frac{1}{2}mv_{\mathrm{i}}^2,$$

$$\Delta K' = \frac{1}{2}mv_{\mathrm{f}}'^2 - \frac{1}{2}mv_{\mathrm{i}}'^2,$$

where i and f stand for "initial" and "final," respectively, and obtain

$$\Delta K' = \Delta K - mv_{0'}\,\Delta v_u, \tag{3}$$

where $\Delta v_u$ is the $u$ component of the velocity change of the object in $S$ frame. We also have $\Delta U' = -W_{\mathrm{c}}'$, which gives

$$\begin{aligned} \Delta U' &= -\int \mathbf{F}_{\mathrm{c}} \cdot d\mathbf{r}' \\ &= -\int \mathbf{F}_{\mathrm{c}} \cdot d\mathbf{r} + v_{0'} \int \mathbf{F}_{\mathrm{c}} \cdot \hat{u}\, dt \\ &= \Delta U + v_{0'} \int F_{cu} dt, \end{aligned} \tag{4}$$

where $F_{cu}$ is the $u$ component of the total conservative force $\mathbf{F}_{\mathrm{c}}$. Considering the mechanical energy change $\Delta E' = \Delta K' + \Delta U'$ in $S'$ frame and using Eqs. (3) and (4), we write

$$\Delta E' = \Delta E + v_{0'}\left(\int F_{cu} dt - m\Delta v_u\right). \tag{5}$$

Equation (5) clearly shows that, in general, $\Delta E'$ is not equal to $\Delta E$.

For constant conservative forces, such as the gravitational force near the surface of Earth, after carrying out the integration over a time interval $\Delta t$, we obtain

$$\Delta E' = \Delta E + v_{0'}(F_{cu}\Delta t - m\Delta v_u). \tag{6}$$

### *Example (a): A book in an elevator*

Consider a book on a table in an elevator moving upward at a constant velocity $v_{0'}$. After a time interval $\Delta t$ the book travels a distance $h = v_{0'}\,\Delta t$, according to an observer A at rest on the ground. Show that Eq. (6) gives the correct answer.

***Solution:*** According to observer A, $\Delta K = 0$, $\Delta U = mgh$, and therefore $\Delta E = mgh$. Observer B at rest with respect to the elevator finds that $\Delta K' = 0$, $\Delta U' = 0$, and so $\Delta E' = 0$. These values are in agreement with Eq. (6), because $F_{cu} = -mg$, $\Delta v_u = 0$, and according to Eq. (6) we have

$$\begin{aligned} \Delta E' &= \Delta E + v_{0'}(-mg\Delta t) \\ &= mgh - mgv_{0'}\Delta t = mgh - mgh = 0. \end{aligned} \tag{7}$$

## Special Cases for $\Delta E = W_{\mathrm{nc}} = 0$

According to Eq. (2) if $W_{\mathrm{nc}} = 0$, we obtain $\Delta E = 0$. Therefore, conservation of mechanical energy can be applied in frame $S$ and according to Eq. (6) we have

$$\Delta E' = v_{0'}(F_{cu}\Delta t - m\Delta v_u). \tag{8}$$

Another useful expression relating $\Delta E$ and $\Delta E'$ is derived using Newton's second law $F_{cu} + F_{\mathrm{nc}u} = mdv_u/dt$, where $F_{cu}$ and $F_{\mathrm{nc}u}$ are the $u$ components of $\mathbf{F}_{\mathrm{c}}$ and $\mathbf{F}_{\mathrm{nc}}$. Thus, we have $(F_{cu} + F_{\mathrm{nc}u})dt = mdv_u$, which gives

$$\int F_{cu} dt - m\Delta v_u = -\int F_{\mathrm{nc}u}\, dt. \tag{9}$$

If we substitute the expression for $\int F_{cu} dt - m\Delta v_u$ from Eq. (9) into Eq. (5), we find

$$\Delta E' = \Delta E - v_{0'} \int F_{\mathrm{nc}u}\, dt. \tag{10}$$

If mechanical energy is conserved in $S$ frame, we obtain

$$\Delta E' = -v_{0'} \int F_{\mathrm{nc}u}\, dt. \tag{11}$$

## Case 1: $\Delta v_u = 0$

If either both initial and final velocities of the object are zero or their $u$ components are equal, Eq. (8) reduces to

$$\Delta E' = v_{0'} F_{cu} \Delta t. \tag{12}$$

### *Example (b): Free fall*

A Ping-Pong ball of mass $m$ is released from rest from a height $h$ above the ground, hits the floor, bounces up, and reaches its initial height, so the initial speed and final speed in the S frame are both 0. Find $\Delta E'$ with respect to observer B moving upward at constant velocity $v_{0'}$ with respect to the ground.

***Solution:*** According to observer B, $\Delta K' = 0$ and $\Delta U' = -mgd$, where $d = v_{0'}\,\Delta t$ is the distance traveled by observer B during the time interval $\Delta t$. So, $\Delta E' = \Delta U' = -mgv_{0'}\,\Delta t$, when computed directly from the definitions. The same result is obtained from Eq. (12),

$$\Delta E' = v_{0'} F_{cu} \Delta t = v_{0'}(-mg)\Delta t. \qquad (13)$$

## Case 2: $\int F_{ncu}\, dt = 0$

Equation (11) shows that if either the total nonconservative force is perpendicular to the velocity of $S'$ with respect to $S$, or $\int F_{ncu}\, dt = 0$, we obtain

$$\Delta E' = 0. \qquad (14)$$

### *Example (c): A crate of mass* m *sliding down a frictionless ramp*

A crate of mass $m$ is sliding down a ramp. The length of the ramp is $d$ and it is inclined at an angle of $\theta$. Frame of reference $S$ is the rest frame of the ramp. $S'$ frame is moving at the final velocity of the crate relative to $S$. Is $\Delta E'$ in agreement with Eq. (14)?

***Solution:*** Note that the normal force is perpendicular to the velocity of $S'$ frame relative to $S$, so $F_{ncu} = 0$. When the crate starts sliding, $v_i = 0$ and $v'_i = -v_f$. Also, the final velocities with respect to $S$ and $S'$ frames are $v_f$ and $v'_f = 0$, respectively. Kinetic and potential energy changes are

$$\Delta K = \frac{1}{2}mv_f^2,\ \Delta K' = -\frac{1}{2}mv_f^2,$$

$\Delta U = -mgd \sin\theta$, and $\Delta U' = mgd \sin\theta$.

The work done by the nonconservative normal force in frame $S$ is zero, so $\Delta E = 0$. Therefore,

$$\frac{1}{2}mv_f^2 = mgd \sin\theta,$$

which leads to $\Delta E' = 0$. This result is in agreement with Eq. (14). It is also in agreement with Eq. (6) if the terms in the parentheses cancel each other. With $\Delta E = 0$, $F_{cu} = mg \sin\theta$, $\Delta t = v_f/(g \sin\theta)$, and $\Delta v_u = v_f$, Eq. (6) results in

$$\Delta E' = v_f(mg\sin\theta\Delta t - mv_f) = mv_f(v_f - v_f) = 0. \qquad (15)$$

## Case 3: $F_{cu} = 0$

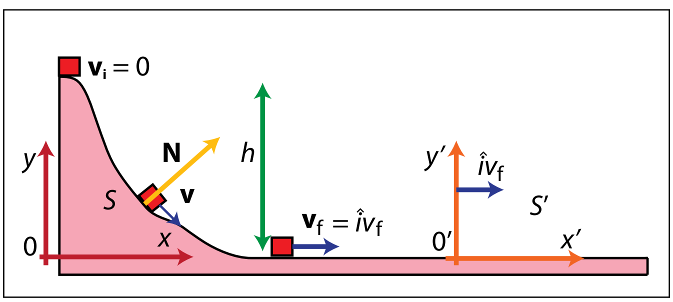


**Fig. 2. A block of mass *m* is released from the top of a curved plane and slides down the track and on a horizontal plane without frictional force. Vertical displacement and final velocity are *h* and $\mathbf{v}_f = \hat{\imath}v_f$, respectively. In the inertial frame *S*′, the initial and final velocities of the body are $\mathbf{v}_i' = -\hat{\imath}v_f$ and $\mathbf{v}_f' = 0$, respectively.**

If the $u$ component of the total conservative force in Eq. (8) is zero, we have

$$\Delta E' = -mv_{0'}\Delta v_u. \qquad (16)$$

### *Example (d): Curved frictionless inclined plane*

A block of mass $m$ is released from the top of a curved frictionless inclined plane and reaches a final velocity. The body continues sliding without friction on a horizontal plane at the constant velocity. In the laboratory inertial reference frame, where the curved inclined plane is at rest, the mechanical energy of the object is constant. However, in another inertial frame moving at the same velocity as the final velocity of the block on the horizontal surface, it is not constant. Show that this is in agreement with Eq. (16).

***Solution:*** According to Fig. 1, the block released from rest at the top of the curved inclined plane slides down the plane and on a horizontal plane. Ignoring the frictional forces, the final velocity of the body $\mathbf{v}_f = \hat{\imath}v_f$ remains constant as the block keeps moving horizontally. The only forces acting on the body are the normal force $\mathbf{N}$ exerted by the curved inclined plane, which acts perpendicular to the track, and the conservative force of gravity, which acts vertically downward. In the laboratory inertial frame of reference $S$, mechanical energy $E$ is conserved. If we consider the zero level of the gravitational potential energy at $y = 0$, according to Fig. 1, we have $U = mgy$.

Now, applying the conservation of mechanical energy to the system gives the final speed $v_f = \sqrt{2gh}$ . In the inertial frame $S'$, the initial mechanical energy is

$$E_i' = \frac{1}{2}mv_f^2 + mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}mv_f^2 = mv_f^2,$$

while when the block is at rest in $S'$, we obtain the final mechanical energy $E'_f = 0$. Therefore, mechanical energy change in the inertial frame $S'$ is

$$\Delta E' = -mv_f^2. \qquad (17)$$

Using the nonconservative work-mechanical energy theorem in the reference frame $S$, we write

$$\Delta E = W_N = \int \mathbf{N}\cdot d\mathbf{r} = \int \mathbf{N}\cdot\mathbf{v}\, dt. \qquad (18)$$

The inner product of the normal force and velocity of the block on any fixed arbitrary surface in $S$ is zero. So conservation of mechanical energy is concluded. Applying the same theorem in $S'$ gives

$$\Delta E' = W_N' = \int \mathbf{N}\cdot\mathbf{v}'\, dt, \qquad (19)$$

where $\mathbf{v}' = \mathbf{v} - \hat{\imath}v_f$ is the velocity of the block in the inertial frame of reference $S'$. Thus, $\mathbf{N}\cdot\mathbf{v}' = -N_x v_f$ and we obtain

$$\Delta E' = -v_f \int_0^{t_f} N_x\, dt. \qquad (20)$$

The gravitational force has no component along the $x$-axis. Therefore, the horizontal acceleration is caused by the $x$-component of the normal force. So, $N_x = m dv_x/dt$ and we have

$N_x dt = m dv_x$. Making this substitution for $N_x dt$ in Eq. (20) results in

$$\Delta E' = -v_{\mathrm{f}} \int_0^{v_{\mathrm{f}}} m\, dv_x = -m v_{\mathrm{f}}^2, \tag{21}$$

which is in agreement with Eq. (17).

Three more examples are provided for interested readers in an online-only appendix.[8]

## Conclusions

The nonconservative work-mechanical energy theorem defined by $\Delta E = W_{\mathrm{nc}}$ is covariant under the Galilean transformations. The mechanical energy of a system can be conserved in a given inertial frame of reference, and not be conserved in another inertial reference frame. In the absence of the nonconservative forces, the mechanical energy is conserved in all inertial reference frames. We obtained simple relations between the change in mechanical energies $\Delta E$ and $\Delta E'$ and clarified the expressions with some examples. Our results can be applied to either particles or extended objects with the same initial and final rotational kinetic and potential energies. Study of mechanical systems with rotational kinetic and potential energy changes under Galilean transformations is suggested for subsequent research. Investigation of the relations between $\Delta E$ and $\Delta E'$ for variable conservative forces in inertial reference frames and also constant conservative forces in noninertial frames of reference is also suggested. Considering Lorentz transformations is another choice for research.

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**Saeed Ghanbari** *is an assistant professor of physics at the University of Zanjan in Zanjan, Iran. He received his PhD in physics in 2008 from the Swinburne University of Technology in Melbourne, Australia. He was a cofounder of the student branch of the Iranian Physical Society in 1993. Since 1994, he has taught almost all undergraduate and many postgraduate courses to physics students. His research interests span permanent magnetic lattices for ultracold atoms, quantum simulation, and physics education. He spends most of his time studying and teaching physics, training for singing, and playing volleyball.* **sghanbari@znu.ac.ir; https://www.znu.ac.ir/members/ghanbari_saeed**