CARLEMAN'S INEQUALITY OVER PRIME NUMBERS

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Carleman's inequality: For positive real numbers $a_1, \ldots, a_n$, Carleman's inequality [1] asserts that

$$\sum_{k=1}^{n} (a_1 \ldots a_k)^{\frac{1}{k}} \leqslant \mathrm{e} \sum_{k=1}^{n} a_k.$$

The constant e is the best possible. Carleman's inequality can be read as follows

$$\frac{\sum_{k=1}^{n} (a_1 \ldots a_k)^{\frac{1}{k}}}{\sum_{k=1}^{n} a_k} \leqslant \mathrm{e}.$$

[1]T. Carleman, *Sur les fonctions quasi-analytiques*, Conférences faites au cinquième congres des mathématiciens Scandinaves, Helsinki (1923), pp. 181–196.

Question: What is the asymptotic growth of the ratio

$$\frac{\sum_{k=1}^{n}\left(a_1\ldots a_k\right)^{\frac{1}{k}}}{\sum_{k=1}^{n}a_k}$$

for several number theoretic sequences a_k?

Carleman's inequality over prime numbers: As usual we denote by p_k the k-th prime, and we define the sequence $(C_n)_{n\geqslant 1}$ by

$$C_n=\frac{\sum_{k=1}^{n}\left(p_1\ldots p_k\right)^{\frac{1}{k}}}{\sum_{k=1}^{n}p_k}.$$

Asymptotic behaviour of C_n: We denote by A_n and G_n, the arithmetic and geometric means of the prime numbers $p_1, \ldots, p_n$, respectively. It is known [2] that

$$A_n = \frac{p_n}{2} + O(n), \qquad \text{and} \qquad G_n = \frac{p_n}{\mathrm{e}} + O(n).$$

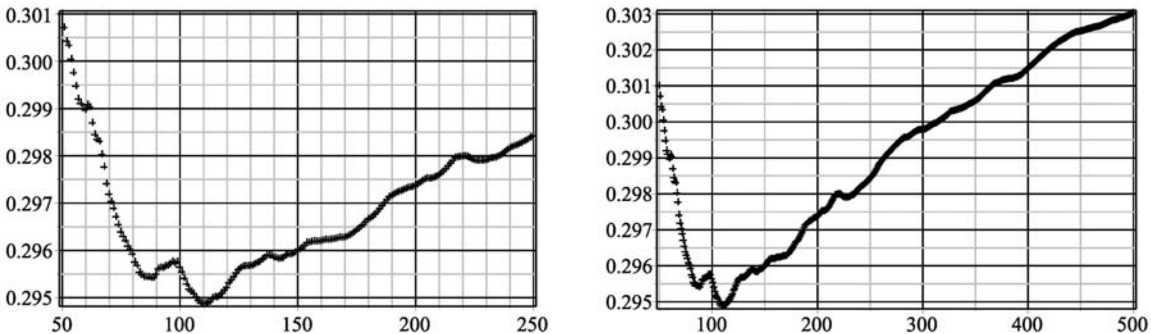
These approximations and the Prime Number Theorem in the form $p_n \sim n \log n$ imply that

$$C_n = \frac{1}{\mathrm{e}} + O\left(\frac{1}{\log n}\right).$$

[2] M. Hassani, *On the ratio of the arithmetic and geometric means of the prime numbers and the number* e, International Journal of Number Theory, 9 (2013), pp. 1593–1603.

Explicit bounds for C_n: By using some explicit bounds concerning A_n, G_n and p_n, for $n \geqslant 2$ we obtain

$$\frac{1}{\mathrm{e}} - \frac{4}{\log n} < C_n < \frac{1}{\mathrm{e}} + \frac{4}{\log n}.$$



Graph of the points (n, C_n) for $50 \leqslant n \leqslant 250$ and $50 \leqslant n \leqslant 500$

Monotonicity of C_n: Computations running over the values of C_n with $1 \leqslant n \leqslant 10^5$ lead us to formulate the following conjecture.

Conjecture. For each $n \geqslant 299$ the sequence C_n is strictly increasing.

If the above conjecture is true, then we obtain

$$\max_{n \geqslant 1} C_n = C_1 = 1, \qquad \text{and} \qquad \min_{n \geqslant 1} C_n = C_{111} \approxeq 0.2949.$$

Also, for $n \geqslant 14$ we have $C_n < \frac{1}{\mathrm{e}}$.

Carleman's inequality over reciprocal of the prime numbers: We define the sequence $(C'_n)_{n\geqslant 1}$ by

$$C'_n = \frac{\sum_{k=1}^{n}\left(\frac{1}{p_1}\cdots\frac{1}{p_k}\right)^{\frac{1}{k}}}{\sum_{k=1}^{n}\frac{1}{p_k}}.$$

Considering the approximations $B_n := \sum_{k=1}^{n}\frac{1}{p_k} = \log\log n + O(1)$ and $\frac{1}{G_k} = \frac{\mathrm{e}}{p_k} + O(\frac{1}{k\log^2 k})$, we get

$$C'_n = \frac{\sum_{k=1}^{n}\frac{1}{G_k}}{B_n} = \frac{\mathrm{e}B_n + O(1)}{B_n} = \mathrm{e} + O\left(\frac{1}{\log\log n}\right).$$

Carleman's inequality over the values of arithmetical functions: We have studied Carleman's inequality over the values of arithmetical functions, more precisely, over the values of Euler function φ, sum of divisors function σ, and their reciprocals[3]. For each positive arithmetical function f let

$$C_f(n) = \frac{\sum_{k=1}^{n} \left(f(1) \ldots f(k)\right)^{\frac{1}{k}}}{\sum_{k=1}^{n} f(k)}.$$

We prove the following asymptotic results, providing non-trivial limit values $\lim_{n\to\infty} C_f(n)$ for the above mentioned functions.

[3]M. Hassani, *Carleman's inequality over the values of Euler function and sum of divisors function*, submitted.

As $n \to \infty$ we have

$$C_\varphi(n) = \eta_\varphi + \eta_\varphi \frac{\log n}{n} + O\Big(\frac{(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}}{n}\Big),$$

where

$$\eta_\varphi = \frac{\pi^2}{6\,\mathrm{e}} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{1}{p}} \approxeq 0.3388.$$

As $n \to \infty$ we have

$$C_{\frac{1}{\varphi}}(n) = \eta_{\frac{1}{\varphi}} + O\Big(\frac{1}{\log n}\Big),$$

where

$$\eta_{\frac{1}{\varphi}} = \frac{2\mathrm{e}\,\pi^4}{315\zeta(3)} \prod_p \left(1 - \frac{1}{p}\right)^{-\frac{1}{p}} \approx 2.2096.$$

As $n \to \infty$ we have

$$C_\sigma(n) = \eta_\sigma + O\Big(\frac{1}{\log n}\Big),$$

where

$$\eta_\sigma = \frac{6}{\mathrm{e}\,\pi^2} \prod_p \left(1 + \frac{1}{p}\right)^{\frac{1}{p}} \prod_{\substack{p^\alpha \\ \alpha \geqslant 2}} \left(\frac{p^{\alpha+1} - 1}{p^{\alpha+1} - p}\right)^{\frac{1}{p^\alpha}} \approxeq 0.3493.$$

As $n \to \infty$ we have

$$C_{\frac{1}{\sigma}}(n) = \eta_{\frac{1}{\sigma}} + O\Big(\frac{\log\log n}{\log n}\Big),$$

where

$$\eta_{\frac{1}{\sigma}} = \frac{\mathrm{e}\prod_{p}\left(1+\frac{1}{p}\right)^{-\frac{1}{p}}\prod_{\substack{p^\alpha\\ \alpha\geqslant 2}}\left(\frac{p^{\alpha+1}-1}{p^{\alpha+1}-p}\right)^{-\frac{1}{p^\alpha}}}{\prod_{p}\left(1-\frac{(p-1)^2}{p}\sum_{\alpha=1}^{\infty}\frac{1}{(p^\alpha-1)(p^{\alpha+1}-1)}\right)} \approxeq 2.2721.$$