DIVISIBILITY TESTS FOR PRIME NUMBERS

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Abstract. We present a general recursive divisibility test for prime numbers other than 2 and 5, based upon their unit digits.

1. Introduction

Perhaps the first divisibility test that a person learns is that a number is even if and only if its last digit is even. Or perhaps it is that a number is a multiple of 5 if and only if it ends in a 0 or in a 5. For all primes other than these two, however, there is no simple "units digit" test for divisibility. In this note, we present a general recursive divisibility test based upon unit digits. To our knowledge, this general a test is new, although it is likely that special cases are known.

2. Main result

We begin by describing the way in which our formula came about, which could serve as motivation for other basic tests. Our aim is to find a way of going from a number with d digits (in base 10) to one with $d-1$, while preserving divisibility. To write our main result, we need the notation $[x]$, which denote the integer part of the real number x.

Theorem 2.1. *Let p be a prime other than 2 or 5 and n be a positive integer, and let u and r be the units digits of p and n, respectively. Then*

$$p|n \Leftrightarrow p\Big|\left[\frac{n}{10}\right]-ar,$$

where

$$a=\begin{cases}\dfrac{up-1}{10} & \text{if } u\equiv 1 \pmod 4,\\[2ex] \dfrac{(10-u)p-1}{10} & \text{if } u\equiv -1 \pmod 4.\end{cases}$$

2010 *Mathematics Subject Classification.* Primary 11A05; Secondary 11A41.
Key words and phrases. divisibility, prime number, digit.

Furthermore, if $0 < a < p$*, then* a *is unique.*

Proof. Let n be a positive integer and p a prime other than 2 or 5. We write $n = 10q + r$ with $0 \leq r \leq 9$ and $p = 10t + u$ with u =1, 3, 7, or 9 (since once of these is the units digit of p). Our goal is to find a simple expression for an integer x such that $p|n$ if and only if $p|(q + x)$.

First assume that $p|(q + x)$. Then $p|(10q + 10x)$, and since $10q = n - r$, if $10x \equiv r \pmod{p}$, then $p|n$.

Conversely, if $p|n$, since $n = 10q + r$, if $10x \equiv r \pmod{p}$ then $p|10(q + x)$. Then, since p and 10 are relatively prime, $p|(q + x)$. Thus, required x satisfies $10x \equiv r \pmod{p}$ and to find it, since p and 10 are relatively prime then there exists a unique a with $0 < a < p$, such that $10a \equiv -1 \pmod{p}$ or $10(-ar) \equiv r \pmod{p}$. So, we may take $x = -ar$. To determine the exact value of a, since $10a \equiv -1 \pmod{p}$ there exists an integer k such that $10a + 1 = pk$ and since $0 < a < p$ then $\frac{1}{p} < k < 10 + \frac{1}{p}$ holds for each prime p other than 2 or 5, and this with $k \neq 10$ force $0 < k < 10$. Thus we have $pk \equiv 1 \pmod{10}$ with $0 < k < 10$. Now, if $u = 1$ then conditions $pk \equiv 1 \pmod{10}$ and $0 < k < 10$ yield $k = 1$ and consequently $a = \frac{p-1}{10}$. We may do same for the other values of u. Thus, we have established the following result. □

Example 2.2. Let $p = 13$ and $n = 897$. We have $u = 3$, and thus $a = \frac{7p-1}{10} = 9$. Also, $r = 7$. Since then $\left[\frac{n}{10}\right] - ar = 89 - 9 \times 7 = 26$, and clearly $13|26$. So, we conclude that $13|897$.

As we mentioned at the beginning, this is a recursion test, in which (generally) the number of digits gets reduced by 1 (as in our example). But, we note that similar tests can be reduced by k digits at a time, rather than just one. The extension for prime powers is also straightforward, and thus for arbitrary divisors via the fundamental arithmetic.

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