Prime Numbers

Mehdi Hassani

University of Zanjan, Department of Mathematics

Prime Numbers : Distribution of primes among natural numbers

Prime Numbers : From Euclid to Euler

Euclid of Alexandria
325 BC-265 BC

The Fundamental Theorem of Arithmetic
Except for the arrangement of the factors every positive integer greater than 1 can be expressed uniquely as a product of primes.

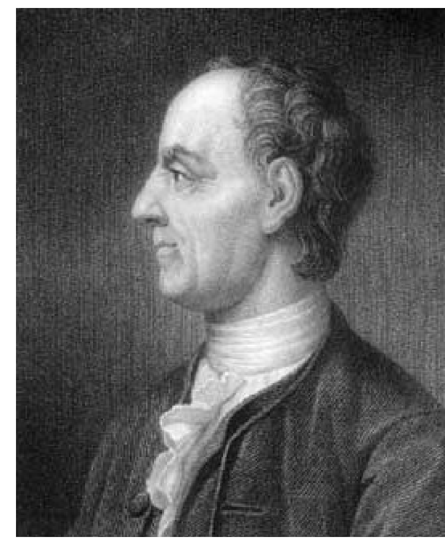
Euclid's Theorem
The number of primes is infinite.

$$\pi(x) = \#\mathbb{P} \cap [2, x] \Longrightarrow \lim_{x\to\infty} \pi(x) = \infty$$

$$p_n = n^{th} \text{ prime} \Longrightarrow \lim_{n\to\infty} p_n = \infty$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{p_n^s}} \Longleftrightarrow \text{The Fundamental Theorem of Arithmetic}$$

$$\lim_{s\to 1^+} \zeta(s) = +\infty \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{p_n} \text{ diverges}$$



Leonhard Euler
1707-1783

Prime Numbers : Legendre's conjecture

$$\pi(x) = \frac{x}{\log x - A(x)} \Longrightarrow \lim_{x\to\infty} A(x) = 1.08366...$$

Adrien-Marie Legendre
1752-1833

Prime Numbers : Gauss' numerical observations and conjecture

Johann Carl Friedrich Gauss
1777-1855

I pondered this problem as a boy, and determined that, at around x, the primes occur with density $1/\ln x$.—C. F. Gauss (letter to Encke, 24 December 1849)

By studying tables of primes, Gauss understood, as a boy of 15 or 16 (in 1792 or 1793), that the primes occur with density $\frac{1}{\log x}$ at around x. In other words

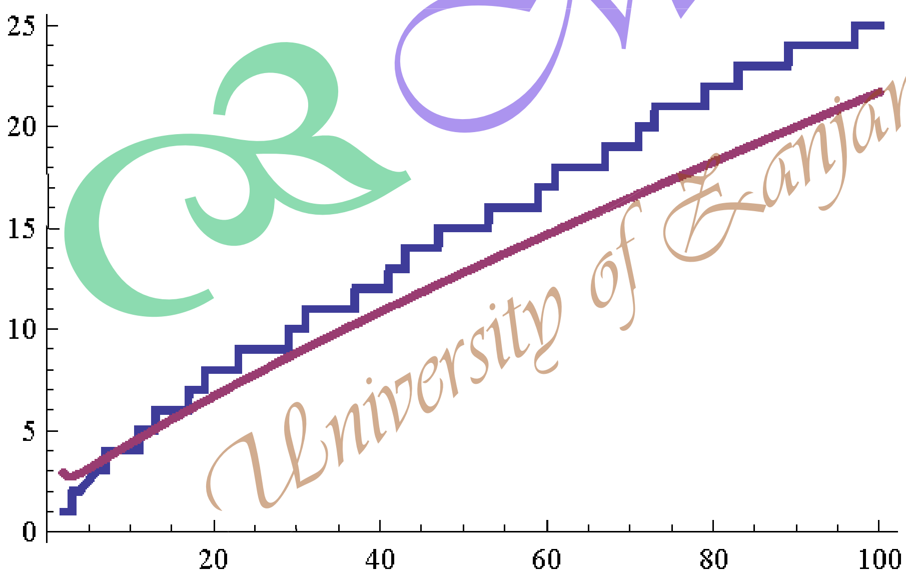
$$\pi(x) := \#\{\text{primes} \le x\} \approx \mathrm{Li}(x) \quad \text{where} \quad \mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

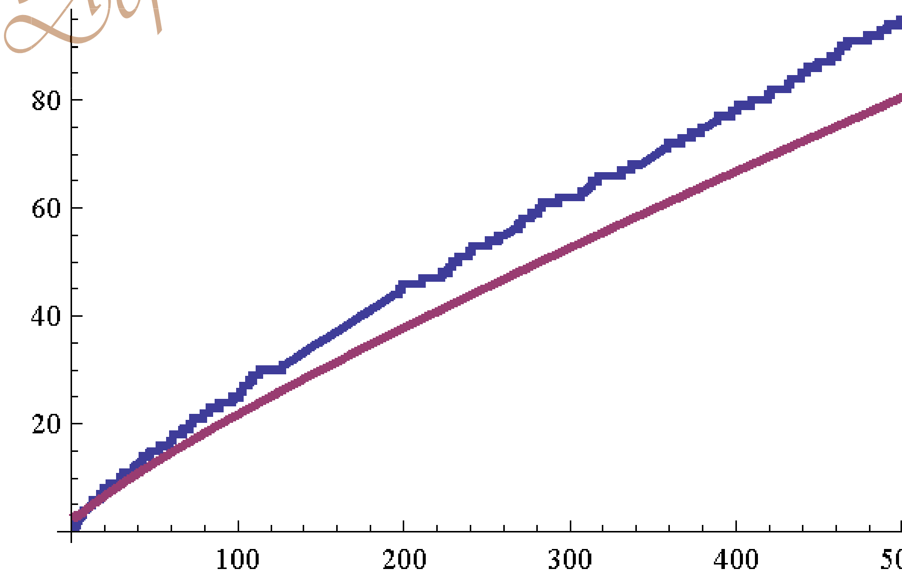
When we integrate by parts we find that a first approximation to $\mathrm{Li}(x)$ is given by $x/(\log x)$ so we can formulate a guess for the number of primes up to x:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

which we write as

$$\pi(x) \sim \frac{x}{\log x}.$$





x	$\pi(x) = \#\{\text{primes} \le x\}$	Overcount: $[\text{Li}(x) - \pi(x)]$
10^8	5761455	753
10^9	50847534	1700
10^{10}	455052511	3103
10^{11}	4118054813	11587
10^{12}	37607912018	38262
10^{13}	346065536839	108970
10^{14}	3204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	279238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2220819602560918840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207
10^{23}	1925320391606803968923	7250186214

Theorem (Littlewood). *There are arbitrarily large values of x for which $\pi(x) >$ Li(x), that is, for which*

$$\#\{\text{primes} \le x\} > \int_2^x \frac{dt}{\ln t}.$$

So what is the smallest x_1 for which $\pi(x_1) > \text{Li}(x_1)$? Skewes obtained an upper bound for x_1 from Littlewood's proof, though not a particularly accessible bound. Skewes proved in 1933 that

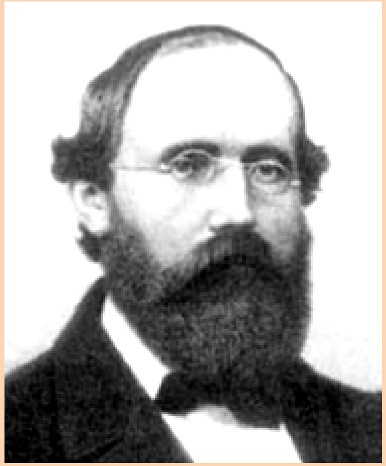
$$x_1 < 10^{10^{10^{10^{34}}}}$$

→ Assuming the Riemann Hypothesis

Skewes (1955):	$x_1 < 10^{10^{10^{10^{1000}}}}$
Lehman (1966):	$x_1 < 2 \times 10^{1165}$
te Riele (1987):	$x_1 < 6.658 \times 10^{370}$
Bays and Hudson (1999):	$x_1 < 1.3982 \times 10^{316}$

Prime Numbers : The Prime Number Theorem

Johann Peter Gustav Lejeune Dirichlet
1805-1859



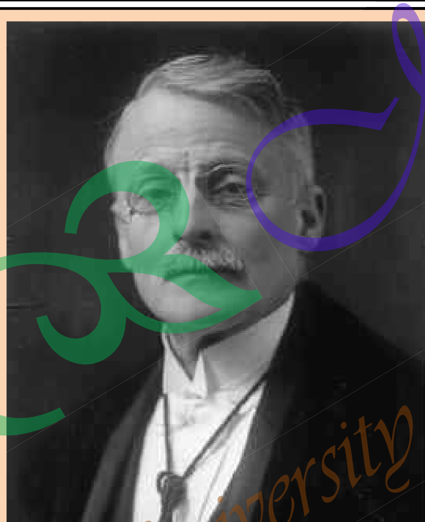
Georg F. Bernhard Riemann
1826-1866

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty)$$

$$\Updownarrow$$

$$p_n \sim n \log n \qquad (n \to \infty)$$

Johann Carl Friedrich Gauss
1777-1855



Charles Jean Gustave Nicolas Baron de la Vallée Poussin
1866-1962

Prime Number Theorem
Born: 1896

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty)$$

Jacques Salomon Hadamard
1865-1963

Prime Numbers : Primes in Arithmetic Progressions

Johann Peter Gustav Lejeune Dirichlet
1805-1859

Theorem (Dirichlet - 1837) *There are infinitely many primes in the progression* $a, a+q, a+2q, \cdots \quad \gcd(a,q)=1$

$$\lim_{s\to 1^+} \zeta(s) = +\infty \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{p_n} \textbf{ diverges}$$

Dirichlet used some analytic properties of the so called L-functions to deduce $\sum_{\substack{p \text{ prime} \\ p\equiv a \pmod q}} \frac{1}{p} = \infty$

$P(a,q)$ denotes the least prime in the arithmetic progression $a \pmod q$ with $\gcd(a,q)=1$

Yuri Vladimirovich Linnik
1915-1972

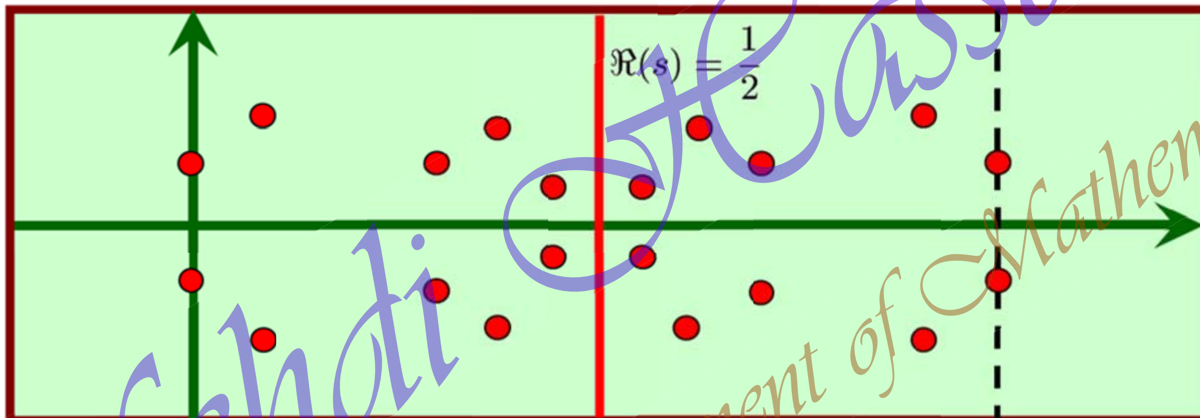
Theorem (Linnik - 1944) *There are absolutely positive constants C and L such that* $P(a,q) < Cq^L$

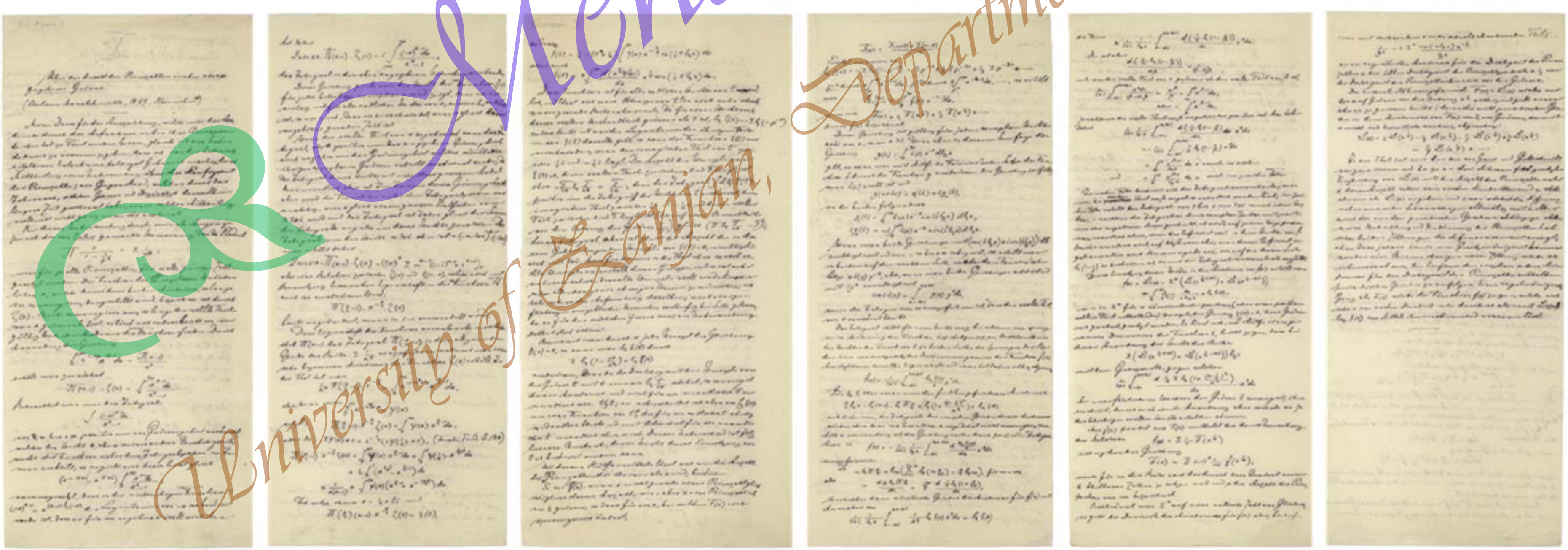
L	Date	Author	L	Date	Author
10000	1957	Pan	36	1977	Graham
5448	1958	Pan	20	1981	Graham
777	1965	Chen	17	1979	Chen
630	1971	Jutila	16	1986	Wang
550	1970	Jutila	13.5	1989	Chen & Liu
168	1977	Chen	8	1991	Wang
80	1977	Jutila	5.5	1992	Heath-Brown

Prime Numbers : Riemann's paper (November 1859)

Georg F. Bernhard Riemann 1826-1866

The *Riemann zeta-function* $\zeta(s)$ is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$



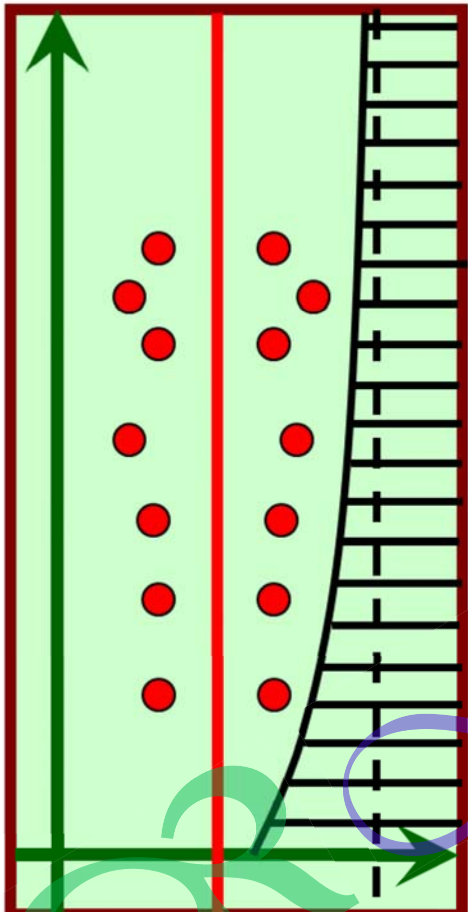


Prime Numbers : Riemann's approximation of primes counting funtion

$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \approx \int_2^x \frac{dt}{\ln t} = \mathrm{Li}(x)$$

$$\pi(x) \approx \mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{1/2}) + \cdots$$

x	$\#\{\text{primes} \le x\}$	Gauss's overcount	Riemann's overcount
10^8	5761455	753	131
10^9	50847534	1700	−15
10^{10}	455052511	3103	−1711
10^{11}	4118054813	11587	−2097
10^{12}	37607912018	38262	−1050
10^{13}	346065536839	108970	−4944
10^{14}	3204941750802	314889	−17569
10^{15}	29844570422669	1052618	76456
10^{16}	279238341033925	3214631	333527
10^{17}	2623557157654233	7956588	−585236
10^{18}	24739954287740860	21949554	−3475062
10^{19}	234057667276344607	99877774	23937697
10^{20}	2220819602560918840	222744643	−4783163
10^{21}	21127269486018731928	597394253	−86210244
10^{22}	201467286689315906290	1932355207	−126677992



$$\sigma \geq 1 - C(\log t)^{-1}$$

$$\pi(x) = Li(x) + O(xe^{-C_1(\log x)^{\frac{1}{2}}})$$

$$\sigma \geq 1 - C(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}$$

$$\pi(x) = Li(x) + O\left(xe^{-C(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}\right)$$

If $\sigma + it$ is a complex number with $0 \le \sigma \le 1$ and $\zeta(\sigma + it) = 0$, then $\sigma = \frac{1}{2}$.