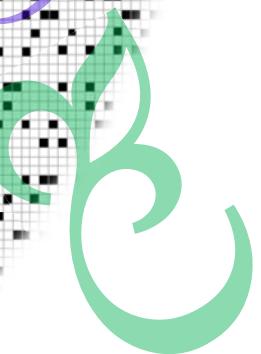


University of
Tunis



Prime numbers

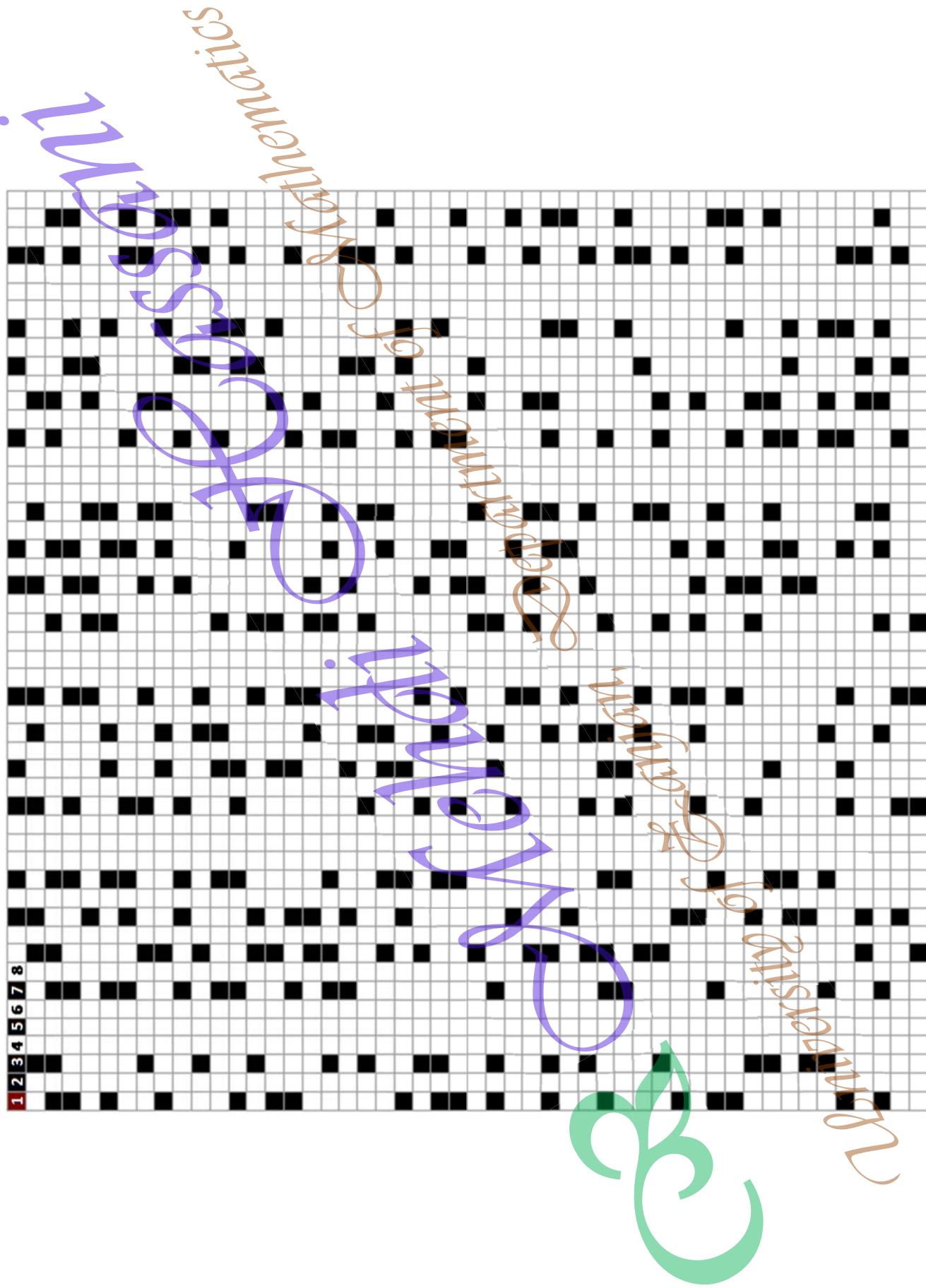
with

Gravitational
Dipole

of Mathematics

and
Physics

Prime Numbers : Distribution of primes among natural numbers



Prime Numbers : From Euclid to Euler



The Fundamental Theorem of Arithmetic

Except for the arrangement of the factors every positive integer greater than 1 can be expressed uniquely as a product of primes.

Euclid's Theorem

The number of primes is infinite.

$$\pi(x) = \#\mathbb{P} \cap [2, x] \rightarrow \lim_{x \rightarrow \infty} \pi(x) = \infty$$

$$p_n = n^{\text{th}} \text{ prime} \rightarrow \lim_{n \rightarrow \infty} p_n = \infty$$

Euclid of Alexandria
325 BC-265 BC



The Fundamental
Theorem
of Arithmetic

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^s}}$$

$$\lim_{s \rightarrow 1^+} \zeta(s) = +\infty \rightarrow \sum_{n=1}^{\infty} \frac{1}{p_n} \text{ diverges}$$

Leonhard Euler
1707-1783

Prime Numbers : Legendre's conjecture

$$\pi(x) = \frac{x}{\log x - A(x)} \quad \Leftrightarrow \quad \lim_{x \rightarrow \infty} A(x) = 1.08366\dots$$



Adrien-Marie Legendre
1752-1833



Prime Numbers : Gauss' numerical observations and conjecture

I pondered this problem as a boy, and determined that, at around x , the primes occur with density $1/\ln x$.—C. F. Gauss (letter to Encke, 24 December 1849)

By studying tables of primes, Gauss understood, as a boy of 15 or 16 (in 1792 or 1793), that the primes occur with density $\frac{1}{\log x}$ at around x . In other words

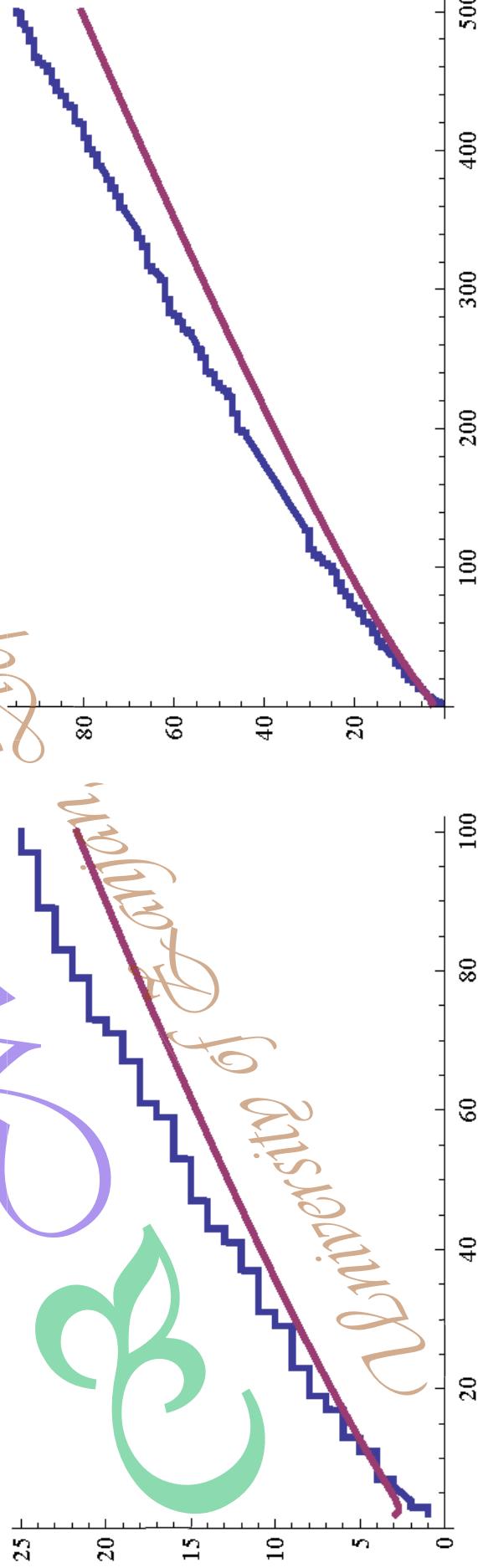
$$\pi(x) := \#\{\text{primes} \leq x\} \approx \text{Li}(x) \quad \text{where} \quad \text{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

When we integrate by parts we find that a first approximation to $\text{Li}(x)$ is given by $x/(\log x)$ so we can formulate a guess for the number of primes up to x :

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x/\log x} = 1,$$

which we write as

$$\pi(x) \sim \frac{x}{\log x}.$$



Johann Carl Friedrich Gauss
1777-1855

Prime Numbers : Skewes number

x	$\pi(x) = \#\{\text{primes} \leq x\}$	Overcount: $[\text{Li}(x) - \pi(x)]$
10^8	5761455	753
10^9	50847534	1700
10^{10}	455052511	3103
10^{11}	4118054813	11587
10^{12}	37607912018	38262
10^{13}	346065536839	108970
10^{14}	3204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	279238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2220819602560918840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207
10^{23}	1925320391606803968923	7250186214

Theorem (Littlewood). There are arbitrarily large values of x for which $\pi(x) > \text{Li}(x)$, that is, for which

$$\#\{\text{primes} \leq x\} > \int_2^x \frac{dt}{\ln t}.$$

So what is the smallest x_1 for which $\pi(x_1) > \text{Li}(x_1)$? Skewes obtained an upper bound for x_1 from Littlewood's proof, though not a particularly accessible bound. Skewes proved in 1933 that

$$x_1 < 10^{10^{10^{10^{34}}}}$$

Skewes (1935):	$x_1 < 10^{10^{10^{10^{1000}}}}$
Lehman (1966):	$x_1 < 2 \times 10^{1165}$
te Riele (1987):	$x_1 < 6.658 \times 10^{370}$
Bays and Hudson (1999):	$x_1 < 1.3982 \times 10^{316}$

Assuming the Riemann Hypothesis

Prime Numbers : The Prime Number Theorem



Johann Peter Gustav
Lejeune Dirichlet
1805-1859



Georg F. Bernhard
Riemann
1826-1866

$$\pi(x) \sim \frac{x}{\log x} \quad (x \rightarrow \infty)$$
$$p_n \sim n \log n \quad (n \rightarrow \infty)$$

↔

Johann Carl Friedrich Gauss
1777-1855



Prime Number Theorem
Born: 1896

$$\pi(x) \sim \frac{x}{\log x} \quad (x \rightarrow \infty)$$

Charles Jean Gustave Nicolas
Baron de la Vallée Poussin
1866-1962

Jacques Salomon
Hadamard
1865-1963

Prime Numbers : Primes in Arithmetic Progressions

Theorem (Dirichlet - 1837) *There are infinitely many primes in the progression $a, a + q, a + 2q, \dots$ such that $\gcd(a, q) = 1$*

$$\lim_{s \rightarrow 1^+} \zeta(s) = +\infty \quad \leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{p_n} \text{ diverges}$$

Dirichlet used some analytic properties of the so called L -functions to deduce that

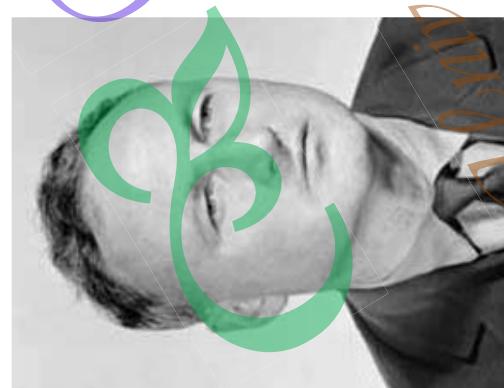
$P(a, q)$ denotes the least prime in the arithmetic progression $a \pmod{q}$ with $\gcd(a, q) = 1$

Theorem (Linnik - 1944) *There are absolutely positive constants C and L such that $P(a, q) < Cq^L$*

L	Date	Author	L	Date	Author
10000	1957	Pan	36	1977	Graham
5448	1958	Pan	20	1981	Graham
777	1965	Chen	17	1979	Chen
630	1971	Jutila	16	1986	Wang
550	1970	Jutila	13.5	1989	Chen & Liu
168	1977	Chen	8	1991	Wang
80	1977	Jutila	5.5	1992	Heath-Brown



Johann Peter Gustav
Lejeune Dirichlet
1805-1859

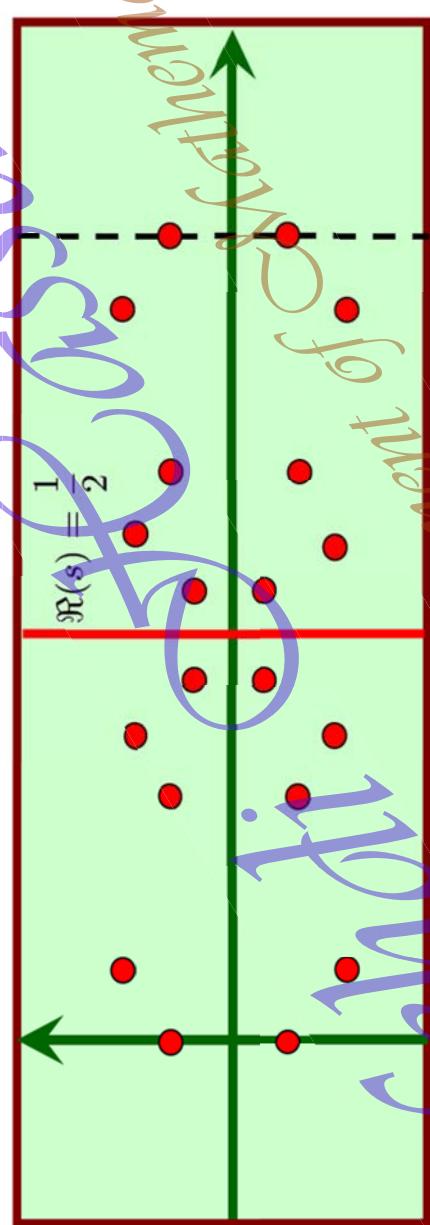


Yuri Vladimirovich Linnik
1915-1972

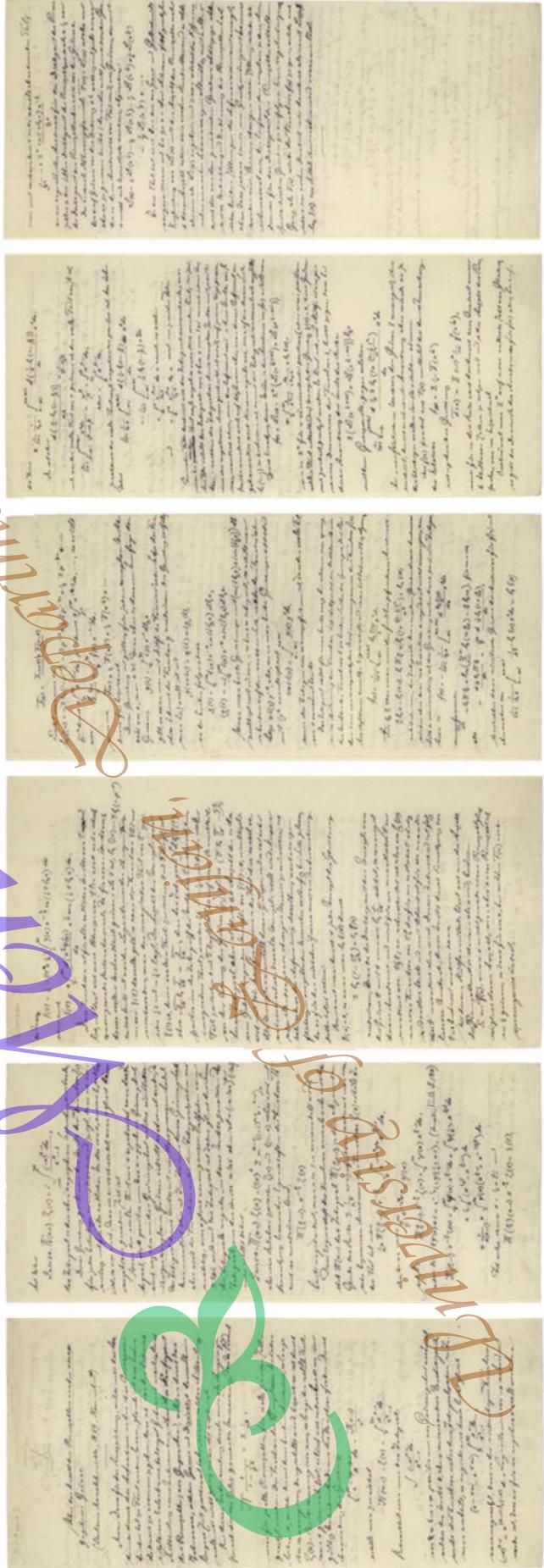
Prime Numbers : Riemann's paper (November 1859)



The **Riemann zeta-function** $\zeta(s)$ is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$



Georg F. Bernhard
Riemann
1826-1866



Prime Numbers : Riemann's approximation of primes counting function

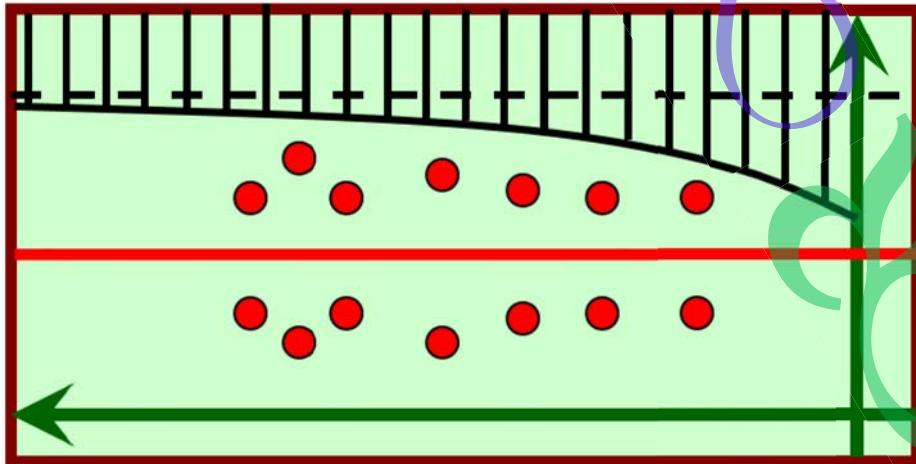
$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \approx \int_2^x \frac{dt}{\ln t} = \text{Li}(x)$$

$$\pi(x) \approx \text{Li}(x) - \frac{1}{2}\text{Li}(x^{1/2}) + \cdots$$

x	# {primes $\leq x$ }	Gauss's overcount	Riemann's overcount
10^8	5761455	753	131
10^9	50847534	1700	-15
10^{10}	455052511	3103	-1711
10^{11}	4118054813	11587	-2097
10^{12}	37607912018	38262	-1050
10^{13}	346065536839	108970	-4944
10^{14}	3204941750802	314889	-17569
10^{15}	29844570422669	1052618	76456
10^{16}	279238341033925	3214631	333527
10^{17}	2623557157654233	7956588	-585236
10^{18}	24739954287740860	21949554	-3475062
10^{19}	234057667276344607	99877774	23937697
10^{20}	2220819602560918840	222744643	-4783163
10^{21}	21127269486018731928	597394253	-86210244
10^{22}	201467286689315906290	1932355207	-126677992

$$\sigma \geq 1 - C(\log t)^{-1}$$

$$\pi(x) = Li(x) + O(xe^{-C_1(\log x)^{\frac{1}{2}}})$$

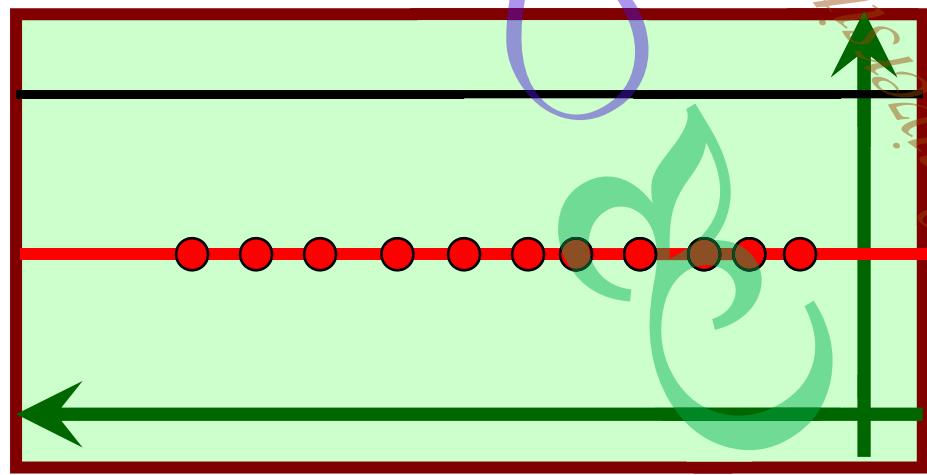


$$\sigma \geq 1 - C(\log t)^{-\frac{2}{3}}$$

$$\pi(x) = Li(x) + O\left(xe^{-C(\log x)^{\frac{3}{5}}(\log \log x)^{-\frac{1}{5}}}\right)$$

Prime Numbers : The Riemann Hypothesis

If $\sigma + it$ is a complex number with $0 \leq \sigma \leq 1$ and $\zeta(\sigma + it) = 0$, then $\sigma = \frac{1}{2}$.



The screenshot shows the homepage of the ZetaGrid website. The header features a large blue banner with the text "Welcome to ZetaGrid" and "The Grid for everybody. Take part, it is a winning game!". Below the banner, the word "Mathematics" is written in a large, stylized, brown font. The main menu includes links for "ZetaGrid", "Acknowledgement", "Performance characteristics", "Riemann Hypothesis", "Prizes", "Motivation", "News", "Statistics", "Software", "Publications", "Forum", and "Links". A sidebar on the right contains the text "What is ZetaGrid? ZetaGrid is a platform independent grid system that uses idle CPU cycles from participating computers. Grid computing can be used for any CPU intensive application which can be split into many separate steps and which would require very long computation times on a single computer. ZetaGrid can be run as a low-priority background process on various platforms like Windows, Linux, AIX, Solaris, HP-UX, and Mac OS X. On windows systems it may also be run in screen saver mode."

ZetaGrid in practice:
At the IBM Development Laboratory in Böblingen ZetaGrid solves one problem in practice, running on six different platforms: The verification of Riemann's Hypothesis is considered to be one of modern mathematics most important problems.