University of Zanjan

Extended Abstracts of the 5th Seminar on Functional Analysis and its Applications

12-13 July 2017

Edited by

S. Maghsoudi

A. Shokri

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ISC Islamic World Science Citation Center

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Edited by: S. Maghsoudi, A. Shokri, A. Morassaei

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Preface

The fifth Seminar on Functional Analysis and its Application was held on 12-13 July 2017 at the University of Zanjan, Iran. This took place in the city of Zanjan located in the north-west of Iran which is known for its beautiful handcrafts such as knives, traditional sandals called Charoogh. The Seminar is a biennial seminar hosted by the Iranian Mathematical Society.

During 2 working days, participants from all over the country presented their recent work in the diverse areas of functional analysis. Based on the reviewers reports, 63 papers were accepted for short presentation and 5 papers in poster section. Four plenary speakers delivered keynote addresses which were welcomed by participants.

On behalf of the organizing committee, I would like to express my heartfelt gratitude to all participants of the Seminar and all who ensured the success of the Seminar. In particular, I should name especially the administration of the University of Zanjan and the Iranian Mathematical Society.

Saeid Maghsoudi

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Short Presentations

STRONG L*-LIMITED PROPERTY IN BANACH LATTICES

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Abstract. Following the concept of L*-limited sets and L*-limited proeprty in Banach spaces, we introduce the concept of almost L*-limited sets and strong L*-limited proeprty in Banach lattices and then we characterize Banach lattices with strong L*-limited proeprty.

1. Introduction

A subset A of a Banach space X is called limited, if every weak* null sequence (x_n^*) in X^* converges uniformly on A, that is,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

If $A \subseteq X^*$ and every weak null sequence (x_n) in X converges uniformly on A, we say that A is an L-set.

We know that every relatively compact subset of X is limited and every limited subset of a dual Banach space is an L-set, but the converse

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Key words and phrases. L–limited set, almost L–limited set, L–limited property.

* Speaker.

of these assertions, in general, are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand–Phillips (GP) property. For example, the classical Banach spaces c_0 and ℓ_1 have the GP property and every separable Banach space, every Schur space (i.e., weak and norm convergence of sequences in the space coincide), and dual of spaces containing no copy of ℓ_1, such as reflexive spaces, have the same property [2]. The reader can find some useful and additional properties of limited and Banach spaces with the GP property in [3].

Recently, the authors in [5], introduced the class of almost L–limited sets and disjoint limited completely continuous (dlcc) operators on Banach lattices. In fact, a bounded linear operator $T : E \to Y$ from a Banach lattice E to a Banach space Y is called dlcc if it carries limited and disjoint weakly null sequences in E to norm null ones in Y. The class of all dlcc operators from X to Y is denoted by $\mathrm{dLcc}(X, Y)$. The authors in [5], characterized these concepts with respect to some well known geometric properties of Banach spaces, such as, GP, reciprocal DP and Grothendieck property.

Here, by the definition of L–limited and L*-limited property, we define almost L*-limited property. It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice is said to be σ–Dedekind complete if its countable subsets that is bounded above has a supremum. A subset A of E is called solid if $|x| \le |y|$ for some $y \in A$ implies that $x \in A$.

Throughout this article, X and Y denote the arbitrary Banach spaces and X^* refers to the dual of the Banach space X. Also E and F denote arbitrary Banach lattices and $E^+ = \{x \in E : x \ge 0\}$ refers to the positive cone of the Banach lattice E and B_E is the closed unit ball of E. If x is an element of a Banach lattice E, then the positive part, negative part and absolute value of x is represented by x^+, x^- and $|x|$, respectively. We refer the reader to [2] and [4] for unexplained terminologies on Banach lattice theory and positive operators.

2. Main results

A subset A of a Banach space X is called an L*-limited set, if every weak null and limited sequence (x_n^*) in X^* converges uniformly on A.

Here, we introduce a new class of sets and operators. Recall that a sequence (x_n) in a Banach lattice E is (pairwise) disjoint, if for each $i \neq j$, $|x_i| \wedge |x_j| = 0$.

Definition 2.1. Let E be a Banach lattice and X be a Banach space. Then

(a) A norm bounded subset B of a dual Banach lattice E^* is said to be an almost L–limited set if every disjoint weakly null and limited sequence (x_n) of E converges uniformly to zero on the set B, that is $\sup_{f\in B} |f(x_n)| \to 0$.
(b) An operator T from a Banach lattice E into a Banach space X is a disjoint limited completely continuous (dlcc) operator if the sequence $(\|Tx_n\|)$ converges to zero for every weakly null and limited sequence of pairwise disjoint elements in E.

Definition 2.2. Let E be a Banach lattice. Then a norm bounded subset B of E is said to be an almost L*-limited set if every disjoint weakly null and limited sequence (f_n) of E^* converges uniformly to zero on the set B, that is, $\sup_{f\in B} |f(x_n)| \to 0$.

It is clear that every DP set in E is almost L*-limited and every subset of an almost L*-limited set is the same. Also, it is evident that every almost L*-limited set is bounded and each L*-limited set is an almost L^*-limited set. The following theorem gives aditional properties of these concepts.

Theorem 2.3. (a) *Every relatively weakly compact subsets of Banach spaces are almost L*-limited.*
(b) *Absolutely closed convex hull of an almost L*-limited set is almost L*-limited.*

Note that the converse of assertion (a) in general, is false. In fact, the following theorem shows that the closed unit ball of c_0 is an L*-limited set and so almost L*-limited, but it is not relatively weakly compact.

As we said before, a Banach lattice E has the positive GP property if each weakly null and limited sequence with positive terms is norm null. It is clear that the GP property implies the positive GP property.

Theorem 2.4. *Let E be a Banach lattice. Then the following are equivalent:*

(a) *E has the positive* GP *property.*
(b) *Every weakly null and disjoint limited sequence in E converges to zero in norm.*

Theorem 2.5. *A Banach space E^* has the positive* GP *property iff every bounded subset of E is an almost* L*-*limited set.*

Definition 2.6. Let X be an arbitrary Banach space and let E be an arbitrary Banach lattice. An operator $T : X \to E$ is called almost L*-limited if $T(B_X)$ is an almost L*-limited set in E. We denote the class of all almost L*-limited operators from X to E by $AL^*_{\text{li}}(X, E)$.

Definition 2.7. A bounded linear operator T from a Banach lattice E into a Banach space Y is dlcc if it carries weakly null and disjoint limited sequences in E to norm null ones.

It is clear that the operator T is almost L*-limited iff T^* is dlcc. Also each weakly compact operator is L*-limited and so almost L*-limited. A Banach space X has the L–limited property, if every L–limited subset of X^* is relatively weakly compact. A Banach space X has the L*-limited property, if every L*-limited set in X^* is relatively weakly compact.

Definition 2.8. A Banach lattice E has the strong L*-limited property, if every almost L*-limited set in E^* is relatively weakly compact.

Theorem 2.9. *For a Banach lattice E, the following are equivalent:*

(a) *E has the strong* L**-limited property.*
(b) *For each Banach space Y, $AL^*_{\text{li}}(Y, E) = W(Y, E)$.*
(c) $AL^*_{\text{li}}(\ell_1, E) = W(\ell_1, E)$.

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ON THE L_p SUBSPACES OF 2π-PERIODIC HOLOMORPHIC FUNCTIONS

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Abstract. Here we obtain isomorphic classes of spaces of 2π-periodic holomorphic functions on the upper half-plane endowed with L_p-norm with respect to a bounded positive non-atomic measure.

1. Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}, G = \{\omega \in \mathbb{C} : \operatorname{Im}\omega > 0\}$ and $\mathbb{T} = \{z \in \mathbb{C} : | z |= 1\}$ denote the open unit disc, upper half-plane and the unit circle in complex plane respectively. Also, let μ be a bounded positive non-atomic measure on $(0,\infty)$ with $\mu((0,\epsilon)) > 0$ for all $\epsilon > 0$. For a holomorphic function $f : G \longrightarrow \mathbb{C}$ and $1 \le p < \infty$ we put

$$\|f\|_{p,\mu}^p = \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} | f(x+it) |^p \, dxd\mu(t)$$

and define spaces

$$H^p_{2\pi,\mu}(G) = \{f : G \longrightarrow \mathbb{C} : f \text{ is } 2\pi \text{ periodic}, \ \|f\|_{p,\mu} < \infty\}.$$

In this paper we intend to obtain isomorphism classification of the spaces $H^p_{2\pi,\mu}(G)$ by using the results of [2] and some well-known results

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Key words and phrases. 2π-periodic holomorphic functions, isomorphism classification, upper half-plane, bounded nonatomic measure.

in the theory of Banach spaces such as boundedness of Riesz projection (see Theorem 2.c.15 of [4]) and the fact that infinite dimensional subspaces of ℓ_p are isomorphic to ℓ_p (see Theorem 2.a.3 of [3]).

In order to mention the main result of this paper we need the following definitions and lemmas.

Definition 1.1. We say that measure μ satisfies condition (B_1) if there exists $n \in \mathbb{N}$ such that

$$\int_0^\infty e^{nt} d\mu(t) = \infty.$$

Example 1.2. (1) Let $n \in \mathbb{N}$ be fixed. Evidently, $\mu : (0, +\infty) \longrightarrow (0, 1/n)$ defined by $\mu(t) = -e^{-nt}/n$ (the distribution function of μ) is a bounded positive non-atomic measure which satisfies condition (B_1).

(2) Define $\mu : (0, +\infty) \longrightarrow (0, 1)$ by $\mu(t) = -te^{-t} - e^{-t}(d\mu(t) = te^{-t})$. Again $\mu(t)$ (the distribution function of μ) is a bounded positive non-atomic measure which satisfies condition (B_1) (here $n = 1$).

(3) Consider the Gaussian function e^{-t^2}. It can be shown that the measure μ $(d\mu(t) = e^{-t^2}dt)$ is a bounded positive non-atomic measure on $(0, \infty)$ which does not satisfy condition (B_1).

Definition 1.3. For each $n \in \mathbb{N}$ we define A_n to be the space of all complex polynomials P with $\deg P \leq n$ endowed with the norm

$$\|P\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mid P(re^{i\varphi}) \mid d\varphi,$$

i.e.,

$$A_n = \{P \mid P : \mathbb{T} \longrightarrow \mathbb{C}, \quad \deg P \leq n, \ \|P\| < \infty\}.$$

Also, we set

$$(\sum_{n\in\mathbb{N}} \oplus A_n)_{(1)} = \{(P_n) : \ P_n \in A_n, \ \sum_{n\in\mathbb{N}} \|P_n\| < \infty\}.$$

Following lemma plays an important role in the proof of Theorm 2.1.

Lemma 1.4. *For each $f \in H^p_{2\pi,\mu}(G)$ there exists $\alpha_k \in \mathbb{C}$ with $f(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ik\omega}$ where the series converges uniformly on compact subsets. Moreover for each $k \in \mathbb{Z}$*

$$\alpha_k = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-ikx+kt} f(x + it) dx \frac{d\mu(t)}{\mu((0, \infty))}.$$

Proof. The proof of the first assertion of the lemma is very similar to the proof of Proposition 2.1 of [1]. For the proofof the last assertion of the lemma we use Hölder's inequality. $\square$

2. Main results

Theorem 2.1. (1) *For $1 < p < \infty$, the Banach space $H^p_{2\pi,\mu}(G)$ is isomorphic to ℓ_p.*

(2) *If measure μ satisfies condition (B_1), then $H^1_{2\pi,\mu}$ is either isomorphic to ℓ_1 or to $(\sum_{n\in\mathbb{N}} \oplus A_n)_{(1)}$.*

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L-DUNFORD-PETTIS SETS AND V-SETS

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ABSTRACT. In this talk, we give some results on the L-Dunford-Pettis sets in Banach spaces. We investigate the relation between L-Dunford-Pettis sets and V-sets with respect to the space of operators.

1. Introduction

In this talk, we denote real Banach spaces with X and Y. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X. A continuous and linear map will be an operator T from X to Y, and the adjoint of T will be denoted by T^*. The set of all operators from X to Y will be denoted by $L(X, Y)$.

An operator $T : X \to Y$ is called completely continuous (or Dunford-Pettis) if T maps weakly null sequences in X into norm convergent sequences in Y. Also, an operator $T : X \to Y$ is called unconditionally convergent if it maps weakly unconditionally convergent series (wuc) in X into unconditionally convergent series (uc) in Y. The set of all unconditionally convergent operators from X to Y will be denoted by $UC(X, Y)$.

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Key words and phrases. L-Dunford-Pettis sets, V-sets, L-sets, relatively compact sets, space of operators.

A bounded subset K of X is called a Dunford-Pettis (DP) subset of X (resp., limited) if

$$\lim_n(\sup\{\mid x_n^*(x)\mid : x \in K\}) = 0$$

for each weakly null (resp., w*-null) sequence (x_n^*) in X^*.

It is well-known that every relatively compact subset of X is limited and every limited subset of X is Dunford-Pettis.

Closely related to the notions of DP-sets and limited sets is the idea of an L-set, see, for instance, Bator [3] and Emmanuele [4, 5]. A Bounded subset K of X^* is called an L-subset of X^* if

$$\lim_n(\sup\{\mid x^*(x_n)\mid : x^* \in K\}) = 0$$

for each weakly null sequence (x_n) in X.

A bounded subset K of X (resp., K of X^*) is called a V*-subset (resp., V-subset of X^*) of X if

$$\lim_n(\sup\{\mid x_n^*(x)\mid : x \in K\}) = 0 \text{ (resp.,} \lim_n(\sup\{\mid x^*(x_n)\mid : x^* \in K\}) = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp., $\sum x_n$ in X).

In [1] and [2], we studied Banach spaces in which every V-set in the dual space is an L-set and Banach spaces in which every V*-set is a Dunford-Pettis set .

Recently, Retbi and Wahbi in [6] introduced the concept of L-Dunford-Pettis sets and L-Dunford-Pettis property in Banach spaces. A norm bounded subset A of the dual space X^* is called an L-Dunford-Pettis set if every weakly null sequence (x_n), which is a Dunford-Pettis set in X converges uniformly to zero on A.

2. Main results

At first, we investigate the relation between V-sets and L-Dunford-Pettis sets in dual spaces.

Theorem 2.1. *Let X be a Banach space. If every* V*-subset of X^* is relatively compact, then every* V*-set in X^* is an* L*-Dunford-Pettis set in X^*.*

The next theorem gives a sufficient condition for a Banach space which every V-set in the dual space be an L-Dunford-Pettis set with respect to the space of unconditionally convergent operators.

Theorem 2.2. *Let X be a Banach space and for every Banach space Y, every unconditionally convergent operator $T : X \to Y$ is completely continuous. Then every* V*-set in X^* is an* L*-Dunford-Pettis set.*

It is well-known that a Banach space X has the Dunford-Pettis relatively compact property (DPrcP) if every Dunford-Pettis subset of X is relatively compact. Certainly, every Schur space has the DPrcP.

For a Banach space X, we showed in [2] that X has the property (MB) and X^* has the DPrcP if and only if every V-subset of X^* is relatively compact.

For Banach spaces X and Y, in [2], we showed that X has the property (MB*) if an operator $T : Y \to X$ is compact whenever $T^* : X^* \to Y^*$ is an unconditionally convergent operator. In the next theorem, we give a chracterization of Banach spaces with the property (MB) which it's dual has the DPrcP.

Theorem 2.3. *Suppose that X is a Banach spaces. Suppose for every Banach space Y, every unconditionally convergent operator $T : X \to Y$ is compact. Then every* V*-subset of X^* is an* L*-set and $\ell_1 \not\hookrightarrow X$.*

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STRONG CONVERGENCE FOR HYBRID PROXIMAL POINT ALGORITHM IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we consider a hybrid projection algorithm for a countable family of mappings of type (P) in Banach spaces. We prove that the generated sequence by the algorithm converges strongly to the common fixed point of the mappings. Furthermore, we apply the result for resolvent of maximal monotone operator for finding a zero of it. The obtained results extend some results in this context.

1. Introduction

One method for approximation of a zero of a maximal monotone operator is the proximal point algorithm. Recently, some authors introduced and studied some modified algorithm of proximal type and thier convergence in Banach spaces; see Li and Song [4], Matsushita and Xu [5], Dadashi and Khatibzadeh [3]. Very recently, Dadashi and Postolache [2] introduced a hybrid proximal point algorithm for special mappings in Banach spaces. They proved strong convergence of the generated sequence by the algorithm in Banach spaces. The mappings in the theorem have to satisfy in a condition, named condition (Z).

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Key words and phrases. maximal monotone operator, proximal point algorithm, resolvent operator in Banach space.

The aim of this paper is to remove condition (Z) and prove an strong convergence theorem. The results can be applied for finding a solution of equilibrium problem and minimizer of a convex function.

2. **Preliminaries**

Let X be a real Banach space. The *normalized duality mapping* J from X into the family of nonempty w^*-compact subsets of its dual X^* is defined by

$$J(x) = \{x^* \in X^* \ : \ \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

for each $x \in X$. Let C be a nonempty closed convex subset of X. For every point $x \in X$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \qquad (y \in C).$$

P_C is called the metric projection of X onto C.

Lemma 2.1 ([1])**.** *Let X be a smooth, strictly convex, and reflexive Banach space, $\{x_n\}$ a sequence in X, and $x \in X$. If $\langle x_n - x, Jx_n - Jx\rangle \to 0$, then $x_n \rightharpoonup x$, $Jx_n \rightharpoonup Jx$, and $\|x_n\| \to \|x\|$.*

Definition 2.2. The multifunction $A : X \to 2^{X^*}$ is called a *monotone operator* if for every $x, y \in X$,

$$\langle x^* - y^*, x - y\rangle \geq 0, \qquad (x^* \in A(x),\ y^* \in A(y)).$$

A monotone operator $A : X \to 2^{X^*}$ is said to be *maximal monotone*, when its graph is not properly included in the graph of any other monotone operator on the same space.

3. Main results

Let X be a smooth Banach space and C a nonempty subset of X. A mapping $T : C \to X$ is said to be of type (P) if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0,$$

for all $x, y \in C$ (see [1]). A mapping $S : C \to X$ is said to be of type (R) if

$$\langle x - Sx - (y - Sy), J(Sx) - J(Sy)\rangle \geq 0.$$

It is easy to see that T is of type (P) if and only if $S = I - T$ is of type (R). The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C \mid x = Tx\}$.

Theorem 3.1. *Let X be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of X and $T_n : C \to X$ a family of mappings of type (P). Then the following hold:*

(1) *If* $\{x_n\}$ *is a bounded sequence in* C *and* $\{T_n x\}$ *is bounded for some* $x \in X$ *or* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, *then* $\{T_n x_n\}$ *is bounded.*
(2) *Assune that the norm of* X *is Fréchet differentiable. If* $\{x_n\}$ *is a sequence in* C *such that* $x_n \to x \in C$ *and* $T_n x \to Tx$, *then* $T_n x_n \rightharpoonup Tx$, $J(x_n - T_n x_n) \rightharpoonup J(x - Tx)$ *and* $\|x_n - T_n x_n\| \to \|x - Tx\|$.
(3) *If* X *has the Kadec-Klee property, then* $T_n x_n \to Tx$.

Theorem 3.2. *Let* X *be a smooth, strictly convex, and reflexive Banach space such that the norm of* X *is Fréchet differentiable,* C *a nonempty closed convex subset of* X *and* $T_n : C \to X$ *a family of mappings of type (P) and* $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$. *Suppose that* T *is a mapping of* C *into* X *defined by* $Tz = \lim\limits_{n\to\infty} T_n z$ *for all* $z \in C$ *such that* $F(T) = F$. *Then the sequence* $\{x_n\}$ *generated by*

$$x_n = P_{C_n}(x), \qquad y_n = T_n(x_n),$$

and

$$C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\},$$

where $C_1 = C$ *and* $x \in X$, *converges strongly to* $P_F(x)$.

Remark 3.3. Let H be a Hilbert space, a mapping $T : C \to H$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle,$$

for all $x, y \in C$. It is obvious that if a mapping $T : C \to H$ is firmly nonexpansive, then

$$\langle Tx - Ty, (x - Tx) - (y - Ty)\rangle \geq 0,$$

holds for all $x, y \in C$ and hence T is of type (P).

Theorem 3.4. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H, $\{T_n\}$ *a sequence of firmly nonexpansive mappings of* C *into* H *such that* $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Consider* T *be a mapping of* C *into* X *defined by* $Tz = \lim\limits_{n\to\infty} T_n z$ *for all* $z \in C$ *such that* $F(T) = F$. *Let* $x \in H$, $\{x_n\}$ *be a sequence in* C *and* $\{C_n\}$ *a sequence of closed convex subsets of* H *defined by* $C_1 = C$ *and*

$$x_n = P_{C_n}(x), \qquad y_n = T_n(x_n),$$

and

$$C_{n+1} = \{z \in C_n : \langle y_n - z, x_n - y_n\rangle \geq 0\},$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $P_F(x)$.

4. **Proximal Point Method**

Definition 4.1. Let X be a smooth, strictly convex, and reflexive Banach space and $A : X \to 2^{X^*}$ be a maximal monotone operator. The operator $J_\lambda : X \to D(A)$ defined by $J_\lambda(x) = x_\lambda$ is called the resolvent of A which x_λ satisfies in $\frac{1}{\lambda}J(x - x_\lambda) \in A(x_\lambda)$.

Theorem 4.2. *Let X be uniformly convex, $A : X \to 2^{X^*}$ be maximal monotone, $F := A^{-1}(0) \neq \emptyset$ and J_{β_n} be the resolvent of A for $\beta_n > 0$. Suppose that the sequence $\{x_n\}$ generated by*

$$y_n = J_{\beta_n}(x_0), \qquad x_n = \alpha_n u + (1 - \alpha_n)y_n + e_n.$$

If $\alpha_n \to 0$, $\beta_n \to \infty$ and $e_n \to 0$, then $x_n \to q = P_F(x_0)$.

Theorem 4.3. *Let X be a smooth, strictly convex, and reflexive Banach space such that the norm of X is Fréchet differentiable. Suppose that $A : X \to 2^{X^*}$ be a maximal monotone operator such that $F := A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence in X and $\{C_n\}$ a sequence of closed convex subset of X defined by $C_1 = X$ and*

$$x_n = P_{C_n}(x), \qquad y_n = J_{\beta_n}(x_n)$$

and

$$C_{n+1} = \{z \in C_n : \ \langle y_n - z, J(x_n - y_n)\rangle \geq 0\},$$

where $x \in X$, $\{\beta_n\} \subset (0, +\infty)$ with $\beta_n \to \infty$ and J_{β_n} is the resolvent of A. Then $\{x_n\}$ converges strongly to the element $P_F(x)$ of F.

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HENIG EFFICIENT SOLUTION FOR VECTOR PARAMETRIC EQUILIBRIUM PROBLEMS

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Abstract. We consider vector parametric equilibrium problems and improve some related theorems for existence of Henig efficient solution for these problems. Also, we provide some applications.

1. Introduction and preliminaries

Equilibrium problems have played a crucial role in the optimization theory. In recent years, as a generalization of some mathematical problems such as the vector variational inequality problems and optimization problems, different types of equilibrium problems for set-valued maps were intensively studied and many results on the existence of solutions for equilibrium problems in different spaces were obtained, see [3, 1, 4].

The outline of this paper is as follows. In this section, we define a vector parametric equilibrium problem, Henig efficient solution for vector parametric equilibrium problem, some preliminary definitions and results which are utilized in the following. In Section 2, we obtain some sufficient conditions for the existence of Henig efficient solution

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of vector parametric equilibrium problems and, finally, we obtain some applications of vector parametric equilibrium problems.

Now, let us recall some definitions and preliminary results which are used in the next sections. Let X, Y, W, and Z be Hausdorff topological vector spaces and Λ, and P be Hausdorff topological spaces. Let A, B and D be nonempty sets of X, W and Z, respectively, and $C : X \times \Lambda \times P \rightrightarrows Y$ be a set-valued mapping such that for any $x \in X$ and for any $\lambda \in \Lambda$, $C(x, \lambda, p)$ is a closed and convex pointed cone in Y such that int $C(x, \lambda, p) \neq \emptyset$. Assume that $e : X \times \Lambda \times P \longrightarrow Y$ is a continuous vector-valued mapping satisfying $e(x, \lambda, p) \in \operatorname{int} C(x, \lambda, p)$. Suppose that $K_1 : A \times \Lambda \longrightarrow 2^A$, $K_2 : A \times \Lambda \longrightarrow 2^B$ and $K_3 : A \times \Lambda \times P \longrightarrow 2^D$ are defined. Let the machinery of the problems be expressed by $F : A \times B \times D \times P \longrightarrow 2^Y$. For any subsets A and B, we adopt the following notations

$$\begin{array}{lll} (u, v)\ r_1\ A \times B & \text{means} & \forall u \in A,\ \forall v \in B, \\ (u, v)\ r_2\ A \times B & \text{means} & \forall u \in A,\ \exists v \in B, \\ (u, v)\ r_3\ A \times B & \text{means} & \exists u \in A,\ \forall v \in B, \end{array}$$

and

$$\begin{array}{lll} \beta_1(A, B) & \text{means} & A \subseteq B, \\ \beta_2(A, B) & \text{means} & A \cap B \neq \emptyset. \end{array}$$

For $r \in \{r_1, r_2, r_3\}$ and $\beta \in \{\beta_1, \beta_2\}$, we consider the following parametric vector quasi-equilibrium problems:

$$(P_{r\beta})\ \forall(\bar{\lambda}, \bar{p}) \in \Lambda \times P \text{ find } \bar{x} \in \operatorname{cl} K_1(\bar{x}, \bar{\lambda}) \text{ such that}$$
$$(y, z)\ r\ K_2(\bar{x}, \bar{\lambda}) \times K_3(\bar{x}, \bar{\lambda}, \bar{p}), \beta\left(F(\bar{x}, y, z, \bar{p}), Y \setminus -\text{int } C(\bar{x}, \bar{\lambda}, \bar{p})\right).$$

We denote the set of the solutions of above problem by $HS_{r\beta}(\bar{\lambda}, \bar{p})$. Let $B(x, \lambda, p)$ be a base of $C(x, \lambda, p)$. Then, $0 \notin \operatorname{cl}(B(x, \lambda, p))$. Let U_Y be the open ball in Y. Let

$$\delta = \sup\{t > 0 :\ (tU_Y) \cap B(x, \lambda, p) = \emptyset\}.$$

Set $V_{B(x,\lambda,p)} = \frac{1}{2}\delta U_Y$. Then $V_{B(x,\lambda,p)}$ is an open convex neighbourhood of $0 \in Y$. We denote cone$\,(B(x, \lambda, p) + U_Y)$ with $C_{U_Y}(B(x, \lambda, p))$. The following definition is a generalization of the definition in [1, 4].

Definition 1.1. $\bar{x} \in \operatorname{cl} K_1(x, \lambda)$ is called a Henig efficient solution of Problem (P_r) if there exists a neighbourhood U_Y of 0 such that

$$(y, z)\ r\ K_2(\bar{x}, \bar{\lambda}) \times K_3(\bar{x}, \bar{\lambda}, \bar{p}) :\ F(\bar{x}, y, z, \bar{p}) \subseteq Y \setminus -\text{int } C_{U_Y}(B(x, \lambda, p)).$$

Special cases of the above problem are considered in [3, 4]. In the sequel, we study special cases of Henig efficient solution of Problem $(P_{r\beta})$.

Remark 1.2. If $C : X \times P \longrightarrow 2^Y$ is a fix map, i.e., for any $x \in X$ and $p \in P$, $C(x, p) = C$ is convex, closed and pointed cone in Y, then the above definition is a generalization of Definition 4.1 in [4] and Definition 4.3 in [2] for optimization problems.

Definition 1.3. A set-valued operator $T : X \longrightarrow 2^Y$ is called:

(a) closed if $Gr(T) = \{(x, y) \in X \times Y : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$,

(b) intersectionally closed on $A \subseteq X$ if

$$\bigcap_{x\in A} \mathrm{cl}\,(T(x)) = \mathrm{cl}\,(\bigcap_{x\in A} T(x)),$$

(c) finite-intersection map (FI-map) iff there exists a topology on X such that:

   (i) the family $(T(y))_{y\in Y}$ has the weak finite intersection property in the sense that $(\mathrm{cl}\,T(y))_{y\in Y}$ has the finite intersection property (i.e., each finite intersection of its sets is nonempty),

   (ii) T is intersectionally closed,

   (iii) $\bigcap_{y\in \bar{N}} \mathrm{cl}\,T(y)$ is compact for some $\bar{N} \in < Y >$.

Theorem 1.4 (Theorem 1 in [3])**.** *Let X be a topological vector space, Y be a nonempty set of X and $T : X \longrightarrow 2^Y$. Then, $\bigcap_{x\in X} T(x) \neq \emptyset$ if and only if T is an FI-map.*

2. Main results

In this section, we obtain some sufficient conditions for existence of Henig efficient solution of vector parametric equilibrium problems. Continue by an idea [5], let us define the set-valued maps $L : \Lambda \longrightarrow 2^X$ as follows $L(\lambda) = (X \setminus \bar{E}(\lambda)) \cup (\bar{E}(\lambda) \setminus \Gamma(\lambda))$, where, $\Gamma(\lambda) = \{x \in X : (y, z)\; r\; K_2(\bar{x}, \bar{\lambda}) \times K_3(\bar{x}, \bar{\lambda}, \bar{p}) :\; F(\bar{x}, y, z, \bar{p}) \subseteq Y \setminus -\mathrm{int}\, C_{U_Y}(B(x, \lambda, p))\}$, and $\bar{E}(\lambda) = \{x \in X :\; x \in \mathrm{cl}\,K_1(x, \lambda)\}$.

Theorem 2.1. *Suppose $L(\lambda)$ is compact or $\bigcap_{x\in N} \mathrm{cl}\,K_1(x, \lambda)$ is compact for some $N \subseteq X \setminus \bar{E}(\lambda)$. Then, there exists a Henig efficient solution for Problem $(P_{r\beta})$.*

The above theorem improves the obtained results in [1, 4].

Now, we will go into further details for three special cases of Problem $(P_{r\beta}(\lambda, p))$ mentioned in the introduction.

(1) Let $T : X \times \Lambda \times P \rightrightarrows L(X, Y)$ be a set-valued map, where $L(X, Y)$ the space of all continuous linear maps of X into Y provided with the pointwise convergence topology, and $g :\; X \times$

$P \longrightarrow X$ and $\eta: X \times X \longrightarrow X$ be such that for all $x \in X$ and $p \in P$, $\eta(x, g(x,p)) = 0$. If for any $x \in X$, $\lambda \in \Lambda$ and $p \in P$ we define $K_1(x,\lambda) = K_2(x,\lambda)$, $K_3(x,\lambda,p) = T(x,\lambda,p)$ and $F(x,y,z,p) = \langle z, \eta(y, g(x,p))\rangle$, then by assuming $r = r_2$ and $\beta = \beta_2$, Problem $(P_{3r\beta}(\lambda,p))$ becomes perturbed vector Stampacchia quasi-variational inequality

$$\bar{x} \in VS_{(\bar{\lambda},\bar{p})}(T, K_1) \Longleftrightarrow \bar{x} \in \operatorname{cl} K_1(\bar{x},\bar{\lambda}):$$
$$\forall y \in K_1(\bar{x},\bar{\lambda}),\ \exists z \in T(\bar{x},\bar{\lambda},\bar{p})$$
$$\langle z, \eta(y, g(\bar{x},\bar{p}))\rangle \notin -\mathrm{int} C(\bar{x},\bar{\lambda},\bar{p}).$$

Furthermore, by assuming $r = r_1$ and $\beta = \beta_1$, and $T : Y \times \Lambda \times P \rightrightarrows L(X,Y)$ with $K_4(y,\lambda,p) = T(y,\lambda,p)$, Problem $(P_{4r\beta}(\lambda,p))$ becomes perturbed vector Minty quasi-variational inequality

$$\bar{x} \in VM_{(\bar{\lambda},\bar{p})}(T, K_1) \Longleftrightarrow \bar{x} \in \operatorname{cl} K_1(\bar{x},\bar{\lambda}):$$
$$\forall y \in K_1(\bar{x},\bar{\lambda}),\ \forall z \in T(y,\bar{\lambda},\bar{p})$$
$$\langle z, \eta(y, g(\bar{x},\bar{p}))\rangle \notin -\mathrm{int}\, C(\bar{x},\bar{\lambda},\bar{p}).$$

(2) If, for all $\bar{x} \in X$, $\bar{p} \in P$ and $\bar{\lambda} \in \Lambda$,

$$K_3(\bar{x},\bar{\lambda},\bar{p}) := K_1(\bar{x},\bar{\lambda}),$$

and for $x, y \in A$, $\lambda \in \Lambda$ and $y \in K_2(x,\lambda)$ we define $F(x,y,z,p) = K_2(z,\lambda) - \{y\}$, then by assuming $r = r_3$ and $\beta = \beta_1$, Problem $(P_{3r\beta}(\lambda,p))$ becomes

$$\exists \bar{x} \in \operatorname{cl} K_1(\bar{x},\bar{\lambda}):\ \exists y \in K_2(\bar{x},\bar{\lambda}) \text{ such that}$$
$$\forall z \in K_1(\bar{x},\bar{\lambda})\ K_2(z,\bar{\lambda}) - \{y\} \subseteq Y \setminus -\mathrm{int}\, C(\bar{x},\bar{\lambda},\bar{p}),$$

which is perturbed optimization problem for set-valued map K_2.

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FIXED POINT THEOREMS FOR A NEW CLASS OF NONLINEAR MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first consider a broad class of nonlinear mappings containing the class of (α,β)-generalized hybrid mappings in Hilbert spaces. Then, we deal with fixed point theorem for these nonlinear mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if C is a bounded closed convex subset of H and $T : C \to C$ is nonexpansive, the set $F(T)$ of fixed points of T is nonempty [1].

Very recently, Kocourek, Takahashi and Yao [2] introduced the following nonlinear mapping: Let C be a nonempty closed convex subset

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* Speaker.

of H. Then, a mapping $T : C \to C$ is said to be (α,β)-generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$.

Theorem 1.1 ([2])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a (α,β)-generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

In this paper, motivated by Kocourek, Takahashi and Yao [2], we introduce a broad class of mappings $T : C \to C$ such that for some $\alpha, \beta, \gamma, m_1, m_2 \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (m_1 - \alpha + \gamma)\|x - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \leq 0$$

for all $x, y \in C$. We call such a mapping an (α, β, γ, m_1, m_2)-generalized hybrid mapping.

Remark 1.2. Let C be a nonempty subset of a Hilbert space and $T : C \to C$. The class of (α, β, γ, m_1, m_2)-generalized hybrid mappings contain many important classes of nonlinear mappings. For example, an (α, β, γ, m_1, m_2)-generalized hybrid mapping is nonexpansive for $(1, 0, 0, 1, 1)$, nonspreading [3] for $(2, 1, 0, 1, 1)$, hybrid [5] for $(3/2, 1/2, 0, 1, 1)$, TY [5] for $(1, 1/2, 0, 1, 1)$ and (α,β)-generalized hybrid [2] for $(\alpha, \beta, 0, 1, 1)$. We can also show that if $x = Tx$ and $0 < m_2 + \gamma \leq m_1 + \gamma$, then for any $y \in C$

$$\alpha\|x - Ty\|^2 + (m_1 - \alpha + \gamma)\|x - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|x - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \leq 0$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that an ($\alpha$, β, γ, m_1, m_2)-generalized hybrid mapping with a fixed point and condition $0 < m_2 + \gamma \leq m_1 + \gamma$ is quasi-nonexpansive [2].

It is well known that the set of fixed points of a quasi-nonexpansive mapping T, is closed and convex [4].

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, ...) \in l^\infty$. Somtimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ . If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, ...) \in l^\infty$,

$$\liminf x_n \le \mu_n(x_n) \le \limsup x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For a proof of existence of a Banach limit and its other elementary, see [1].

Using Banach limits, Takahashi and Yao [5] proved the following fixed point theorem.

Theorem 1.3. *Let H be a Hilbert space. Let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \le \mu_n \|T^n x - y\|^2, \qquad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

2. Main results

In this section, we first prove the following fixed point theorem for (α, β, γ, m_1, m_2)-generalized hybrid mappings in a Hilbert space.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a (α, β, γ, m_1, m_2)-generalized hybrid mapping with condition $0 < m_2 + \gamma \le m_1 + \gamma$. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Proof. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$. So, $\{T^n z\}$ is bounded. We show the reverse. Take $z \in C$ such that $\{T^n z\}$ is bounded, since $T : C \to C$ is a (α, β, γ, m_1, m_2)-generalized hybrid mapping, there are $\alpha, \beta, \gamma, m_1, m_2 \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (m_1 - \alpha + \gamma)\|x - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \le 0$$

for all $x, y \in C$.

Now if μ is a Banach limit, then for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\alpha\|T^{n+1}z - Ty\|^2 + (m_1 - \alpha + \gamma)\|T^n z - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|T^{n+1}z - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|T^n z - y\|^2 \le 0.$$

Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then, we have

$$\mu_n(\alpha\|T^{n+1}z - Ty\|^2 + (m_1 - \alpha + \gamma)\|T^n z - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|T^{n+1}z - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|T^n z - y\|^2) \le 0.$$

So, we obtain

$$\begin{aligned}\alpha\mu_n\|T^{n+1}z - Ty\|^2 + (m_1 - \alpha + \gamma)\mu_n\|T^n z - Ty\|^2 \\ - (\beta + (\beta - \alpha)\gamma)\mu_n\|T^{n+1}z - y\|^2 \\ - (m_2 - \beta - (\beta - \alpha - 1)\gamma)\mu_n\|T^n z - y\|^2 \leq 0\end{aligned}$$

and hence

$$\begin{aligned}\alpha\mu_n\|T^n z - Ty\|^2 + (m_1 - \alpha + \gamma)\mu_n\|T^n z - Ty\|^2 \\ - (\beta + (\beta - \alpha)\gamma)\mu_n\|T^n z - y\|^2 \\ - (m_2 - \beta - (\beta - \alpha - 1)\gamma)\mu_n\|T^n z - y\|^2 \leq 0.\end{aligned}$$

This implies

$$(m_1 + \gamma)\mu_n\|T^n z - Ty\|^2 \leq (m_2 + \gamma)\mu_n\|T^n z - y\|^2$$

for all $y \in C$. So, we obtain, by condition $0 < m_2 + \gamma \leq m_1 + \gamma$,

$$\mu_n\|T^n z - Ty\|^2 \leq \mu_n\|T^n z - y\|^2$$

for all $y \in C$. By Theorem 1.3, we have a fixed point in C. □

As a direct consequence of Theorem 2.2, we have the following result.

Corollary 2.2. *Let C be a nonempty closed convex bounded subset of a Hilbert space H and let $T : C \to C$ be a $(\alpha, \beta, \gamma, m_1, m_2)$-generalized hybrid mapping with condition $0 < m_2 + \gamma \leq m_1 + \gamma$. Then T has a fixed point.*

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FUZZY FRAME ON FUZZY HILBERT SPACES

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Abstract. In this paper, we construct a fuzzy inner product space from classical inner product space on $\mathbb{C}$. We define fuzzy norm in the sense of Bag and Samanta by fuzzy inner product function which is induced from fuzzy inner product. Also, we define fuzzy Hilbert spaces and fuzzy frame in the fuzzy Hilbert spaces.

1. PRELIMINARIES

In this section, some definitions and preliminaries are given which will be used in this paper.

Definition 1.1 ([4])**.** Let U be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : U \times U \times \mathbb{C} \longrightarrow I = [0, 1]$ be a mapping such that the following hold:

(FIP1) For $s, t \in \mathbb{C}, \mu(x + y, z, |t| + |s|) \geq \min\{\mu(x, z, |t|), \mu(y, z, |s|)\}$.
(FIP2) For $s, t \in \mathbb{C}, \mu(x, y, |st|) \leq \min\{\mu(x, x, |s|^2), \mu(y, y, |t|^2)\}$.
(FIP3) For $t \in \mathbb{C}, \mu(x, y, t) = \mu\left(x, y, \bar{t}\right)$.
(FIP4) $\mu(\alpha x, y, t) = \mu(x, y, t/|\alpha|), \alpha (\neq 0) \in \mathbb{C}, t \in \mathbb{C}$.
(FIP5) $\forall t \in \mathbb{C}\backslash\mathbb{R}^+, \mu(x, x, t) = 0$.
(FIP6) $(\forall t > 0, \mu(x, x, t) = 1)$ iff $x = \underline{0}$.

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(FIP7) $\mu(x,x,.) : \mathbb{R} \longrightarrow I$ is a monotonic non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} \mu(\alpha x, x, t) = 1$.

We call μ a fuzzy inner product function on U and (U,μ) is called a fuzzy inner product space (FIP space).

Theorem 1.2 ([4])**.** *Let U be a linear space over $\mathbb{C}$. Let μ be an FIP on U. Then*

$$N(x,t) = \begin{cases} \mu(x,x,t^2) & t \in \mathbb{R},\ t > 0 \\ 0 & t \leq 0 \end{cases}$$

is a fuzzy norm on U.

Now, if μ satisfies the following conditions:

(FIP8) $\forall t > 0, \mu(x,x,t^2) > 0 \Rightarrow x = \underline{0}$.

(FIP9) *For all $x, y \in U$ and $p, q \in \mathbb{R}$,*

$$\mu\left(x+y, x+y, 2q^2\right) \wedge \mu\left(x-y, x-y, 2p^2\right) \geq \mu\left(x,x,p^2\right) \wedge \mu\left(y,y,q^2\right).$$

Then $\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\}$, ($\alpha \in (0,1)$), is an ordinary norm satisfying parallelogram law.

Using Polarization identity, we can get ordinary inner product, called the $\langle .,. \rangle_\alpha$-inner product, for $\alpha \in (0,1)$, as follows:

$$\langle x,y \rangle_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right) + \frac{1}{4}i\left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right).$$

Definition 1.3 ([4])**.** (1) Suppose (U,μ) is an FIP space satisfying (FIP8). Then U is said to be level complete if for any $\alpha \in (0,1)$, every Cauchy sequence converges w.r.t. $\|.\|_\alpha$ (the α-norm generated by the fuzzy norm N which is induced by fuzzy inner product μ).

(2) Let (U,μ) be an FIP space. Then U is said to be a fuzzy Hilbert space if it is level complete.

2. Main results

Example 2.1. Let $(U, \langle .,. \rangle)$ be a real inner product space. Define function $\mu : U \times U \times \mathbb{R} \to [0,1]$ by

$$\mu(x,y,t) = \begin{cases} \dfrac{|t|}{|t| + \|x\|\|y\|} & \text{if } t > \|x\|\|y\| \\ 0 & \text{if } t \leq \|x\|\|y\|. \end{cases}$$

We can show that μ is a fuzzy inner product function and by Theorem 1.2, we have fuzzy norm on U. Since conditions (FIP8) and (FIP9) are satisfied, so we will have a fuzzy Hilbert space.

Definition 2.2. Let (U,μ) be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $\alpha \in (0,1)$. A countable family of elements $\{x_k\}_{k=1}^{\infty}$ in U is a fuzzy frame for U if there exist constants $A, B > 0$ such that for all $x \in U$ and $\alpha \in (0,1)$:

$$A\|x\|_\alpha^2 \leq \textstyle\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 \leq B\|x\|_\alpha^2.$$

The numbers A and B are called frame bounds. They are not unique. The optimal lower frame bound is supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. Note that the optimal frame bounds are actually frame bounds. If $\|x_k\|_\alpha = 1$ then the fuzzy frame is normalized. A fuzzy frame $\{x_k\}_{k=1}^{\infty}$ is tight if we can choose $A = B$ in the definition above and if $A = B = 1$ we call it a Parseval fuzzy frame.

Remark 2.3. The Cauchy-Schwarz inequality shows that

$$\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 \leq \sum_{k=1}^{\infty} \|x_k\|^2\|x\|_\alpha^2 \leq B\|x\|_\alpha^2,$$

i.e., the upper fuzzy frame condition is automatically satisfied.

Theorem 2.4. *Let (U,μ) be a fuzzy Hilbert space satisfying (FIP 8) and (FIP 9) and $\alpha \in (0,1)$ and $\{e_k\}_{k=1}^{\infty}$ be an α-fuzzy orthonormal sequence in U. Then for every $x \in U$,*

$$\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 \leq B\|x\|_\alpha^2.$$

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A CHARACTERIZATION OF RESOLVENT ALGEBRA FOR SOME OPERATORS

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Abstract. We characterize the resolvent subalgebra R_A for a matrix operator A on a finite dimensional Hilbert space $\mathcal{H}$. Then we show that $R_A = R_A^c = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\ \}$. Also in the infinite dimensional case we show that if A is surjective or bounded bellow then $R_A = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\ \}$.

1. Introduction

Suppose that $\mathcal{H}$ is a separable infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. The closed subspace $M \subset \mathcal{H}$ is said to be an invariant subspace of the collection $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, if it is invariant for all $A \in \mathcal{S}$, i.e., $AM \subset M$ for each $A \in \mathcal{S}$. At the same token, M is called a hyperinvariant subspace of A if it is invariant for its *comutant*

$$\{A\}' = \{T \in \mathcal{L}(\mathcal{H}) : TA = AT\}.$$

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* Speaker.

The existence of hyperinvariant subspaces of a bounded linear operator on a Hilbert space is a long-standing problem, and has recently attracted the attention of a great number of mathematicians.

Deddens [1] introduced the algebra

$$B_A = \{T \in \mathcal{L}(\mathcal{H}) : \sup_{m \geq 1} \|A^m T A^{-m}\| < \infty\}$$

for an invertible linear operator $A \in \mathcal{L}(\mathcal{H})$. It is clear that $\{A\}' \subset B_A$, and thus every invariant subspace of B_A is also a hyperinvariant subspace of A.

Assuming that, for each integer $m \geq 1$, the operator $1 + mA$ is invertible, the *resolvent algebra* is

$$R_A = \{T \in \mathcal{L}(\mathcal{H}) : \sup_{m \geq 1} \|(1 + mA)T(1 + mA)^{-1}\| < \infty\}.$$

Clearly $\{A\}' \subset R_A$. In [4], it was shown that $R_A = \{A\}'$, when A is a nilpotent operator. In [2], it was shown that R_A posses nontrivial invariant subspaces, when A is an algebraic operator of degree 2. More properties of R_A for algebraic operators were studied in [3].

To introduce the last algebra that we will consider, let

$$R_m = \left(\sum_{n=0}^{\infty} \frac{A^{*n} A^n}{\left(r(A) + \frac{1}{m}\right)^{2n}} \right)^{1/2},$$

where $r(A)$ is the spectral radius of A. Then define

$$\mathcal{B}_A = \{T \in \mathcal{L}(\mathcal{H}) : \sup_{m \geq 1} \|R_m T R_m^{-1}\| < \infty\}$$

that was defined in [5]. We characterize the elements of R_A .

In [3], it is introduced the resolvent space R_a in some Banach algebra $\mathcal{A}$ as follows;

$$R_a = \{b \in \mathcal{A} : \liminf_{w \to \infty, w \in \mathbb{C}} (1 - wa)b(1 - wa)^{-1} < \infty\}$$

and it is shown that if $\mathcal{A} = \mathcal{L}(\mathcal{X})$ and $A \in \mathcal{L}_1(\mathcal{X})$, then the resolvent spaces

$$R_A = \{T \in \mathcal{L}(\mathcal{X}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\} \tag{1.1}$$

It is easy to see that this fact holds for resolvent algebras. Hence, the resolvent space coincides with resolvent algebras for algebraic operators. In the next section we show that $R_A = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\}$ for some special operators A. We recall that a subspace M reduces an operator A, if M and $M^{\perp}$ are invariant subspaces for T.

2. Resolvent of finite rank operators

Let $\mathcal{H}$ be a finite dimensional space, $\{e_i : i = 1, ..., n\}$ be the standard basis of $\mathcal{H}$ and $A = (a_{ij})$ be a matrix operator on $\mathcal{H}$ with respect to $\{e_i : i = 1, ..., n\}$. In this case we give the following lemma:

Lemma 2.1. *Let $A = (a_{ij})$ and assume that $I+mA$ is invertible. Then for each $i = 1, \ldots, n$*

$$(1+mA)^{-1}e_i \quad = \quad \sum_{i=1}^{n} \alpha_{ij} e_j, \tag{2.1}$$

where

$$\alpha_{ij} \quad = \quad \frac{M_{ij}}{\det(b_{ij})}.$$

and M_{ij} is the cofactor of the matrix

$$(b_{ij}) = (\delta_{ij} + m a_{ij}).$$

Using the above lemma, in the finite dimensional case, we have the following theorem:

Theorem 2.2. *Let $A = (a_{ij}) \in \mathcal{L}(\mathcal{H})$. Then*

$$R_A = R_A^c = \big\{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\big\}.$$

Now we consider the resolvent algebra of invertible, surjective and bounded bellow operators on an infinite dimensional Hilbert space. Note that if $M \subset \mathcal{H}$ and $\mathcal{L}(\mathcal{H})$, then using $\mathcal{H} = M \oplus M^{\perp}$, we can write T as follows

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

where $T_1 : \mathcal{N}(A) \to \mathcal{N}(A), T_2 : \mathcal{N}(A)^{\perp} \to \mathcal{N}(A), T_3 : \mathcal{N}(A) \to \mathcal{N}(A)^{\perp}$, and $T_4 : \mathcal{N}(A)^{\perp} \to \mathcal{N}(A)^{\perp}$.

The following theorems show that (1.1) is valid for surjective or bounded bellow operators.

Theorem 2.3. *Let A be a surjective operator on $\mathcal{H}$. Then*

$$R_A = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\}.$$

Theorem 2.4. *Let A be a bounded below operator on $\mathcal{H}$. Then we have*

$$R_A = \mathcal{L}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{N}(A) \in \mathrm{Lat}(T)\}.$$

Proof. Let A be a bounded below operator then there exists $\gamma > 0$ such that $\|Ax\| \geq \gamma\|x\|$, for any $x \in \mathcal{H}$. Invertibility of $1+mA$ implies that $(1+mA)^{-1}A = \frac{1-(1+mA)^{-1}}{m}$. So, for any $x \in \mathcal{H}$ with $\|x\| = 1$ we have

$$\gamma\|(1+mA)^{-1}x\| \leq \|A(1+mA)^{-1}x\| \leq \frac{1+\|(1+mA)^{-1}x\|}{m}.$$

Hence

$$\|(1+mA)^{-1}\| \leq \frac{1}{\gamma m - 1},$$

therefore $\sup_m \|(1+mA)T(1+mA)^{-1}\| < \infty$. $\square$

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A COMMON FIXED POINT THEOREM FOR UPPER SEMI-CONTINUOUS MAPPINGS

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Abstract. In this paper we present a common fixed point theorem for a family of self set-valued mapping on a paracompact subset of a locally convex topological vector space. Some examples are given in order to illustrate and show the main theorem improves the corresponding results in this area. Furthermore, some applications of the main theorem are stated.

1. Introduction

It is well known that many problems in topology, nonlinear analysis, mathematical economics, and game theory give rise to fixed point problems for some uni-valued or set-valued mappings. The fixed point theory of set-valued mappings began in 1973 [1]. In some recent papers, for example [5], by using some new concepts of generalized KKM mappings, the authors establish common fixed point theorems for families of set-valued mappings in Hausdorff topological vector spaces. The main result of this paper fits in this group of results. In the last part of the paper, there is an analogue of Theorem 4.1 [4] with less condition.

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* Speaker.

Let X be a topological space, we briefly recall some known definition.

Definition 1.1. A cover of X is a collection of X whose union contains X. In symbols, if $U = \{U_i : i \in I\}$ is an indexed family of subsets X, then U is a cover of X, if $X \subseteq \bigcup_{i \in I} U_i$. A cover is open if all its members are open sets.

Definition 1.2. A refinement of a cover of X is a new cover of the same space such that every set in the new cover is a subset of some set in the old cover. In symbols, the cover $V = \{V_j : j \in J\}$ is a refinement of the cover $U = \{U_i : i \in I\}$ if and only if, for any V_j in V, there exist some U_i in U such that V_j is contained in U_i.

Definition 1.3. An open cover of X is locally finite if every point of the space has a neighborhood that intersects only finitely many sets in the cover. In symbols, $U = \{U_i : i \in I\}$ is locally finite if and only if, for any $x \in X$ there exists some neighbourhood $V(x)$ of x such that the set $\{i \in I : U_i \bigcap V(x) \neq \emptyset\}$ is finite

Definition 1.4 ([6])**.** X is said to be paracompact if every open cover has a locally finite open refinement.

Example 1.5. Every compact space is paracompact.

Theorem 1.6 ([6])**.** *Every metric space is paracompact.*

Definition 1.7. Let X be a topological space and let $\{U_i\}_{i \in I}$ be an open cover of X. A partition of unity relative to the cover $\{U_i\}_{i \in I}$ consist of a collection of continuous functions $f_i : X \to [0, 1]$ such that:

(i) for every function $f_i : X \to [0, 1]$ from the collection, there is an open set from the cover such that the support of f_i is contained in U_i.

(ii) for every point $x \in X$ there is a neighborhood W_x of x such that all but finitely many of the functions in the collecion are identically 0 in W_x.

(iii) $\sum_{i \in I} f_i(x) = 1$.

Theorem 1.8 ([6])**.** *Every paracompact Hausdorff space admits partions of unity subordinate to any open cover.*

Definition 1.9. A multifunction $T : X \rightrightarrows Y$ between topological spaces is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $T(x) \bigcap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $T(t) \bigcap V \neq \emptyset$; Φ is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) **continuous** on X if it is at the same time u.s.c. and l.s.c. on X.

(v) **closed** if the graph $G_r(T)$ of T, i.e., $\{(x, y) : x \in X, y \in T(x)\}$, is a closed set in $X \times Y$.

(vi) **compact** if the closure of range T, i.e., $\overline{T(X)}$, is compact, where $T(X) = \cup_{x \in X} T(x)$.

Theorem 1.10. *For a multifunction $T : X \rightrightarrows Y$ between topological spaces and a point $x \in X$ the following statements are equivalent:*

1) *The multifunction T is upper semicontinuous at x and $T(x)$ is compact.*
2) *For any net $\{x_i\} \subseteq X$ such that $x_i \to x$ and for every $y_i \in T(x_i)$, there exist $y \in T(x)$ and a subnet $\{y_j\}$ of $\{y_i\}$ such that $y_j \to y$.*

Theorem 1.11 (Himmelberg)**.** *If X be a nonempty convex subset of a locally convex space X and $T : X \rightrightarrows X$ is an upper semicontinuous multifunction with nonempty closed convex values and compact, then T has a fixed point.*

The following lemma is an application of a minimax theorem.

Lemma 1.12. *Let D be a compact convex and K be a nonempty convex subset of topological vector space X and $P : D \times K \to R$ be a concave and upper semicontinuous in the first variable and convex in the second variable such that*

$$\max_{\xi \in D} P(\xi, y) \geq 0 \quad (y \in K),$$

then there exists $\bar{\xi} \in D$ such that

$$P(\bar{\xi}, y) \geq 0 \quad (y \in K).$$

2. Main results

In this section we are going to present a common fixed point theorem for a family of self set-valued mapping on a paracompact subset of a locally convex topological vector space. Moreover, some examples to illustrate the main result are given. This result can be view as a generalization of main result in [4].

Theorem 2.1. *Suppose that X be a non-empty, paracompact, convex subset of a Hausdorff locally convex topological vector space E. Let $\{T_i : X \rightrightarrows X\}_{i\in I}$ be a family of nonempty, convex set valued mapping which are compact and upper semicontinuous. If for any $x \in X$ and each nonempty finite family of indices $J \subseteq I$*

$$x \notin [\operatorname{conv}(\bigcup_{i\in J} T_i(x)) - (\bigcup_{i\in J} T_i(x))],$$

then the mapping T_i, $i \in I$, have a common fixed point.

Example 2.2. Let $X = \mathbb{R}_+$ and $\{T_i : i \in I\}$ be a family indexed by a set I, of set-valued mapping on X such that $T_i(x) = [0, x]$, then the mapping T_i, $i \in I$, have a common fixed point. Not that $\mathbb{R}_+$ is not compact.

Example 2.3. Let $X = \mathbb{R}_+$ and $T_n(x) = [0, nx]$, for $n \in \mathbb{N}$, then the mapping T_n, $n \in \mathbb{N}$, have a common fixed point.

Now, let X be a compact and convex subset of a locally convex Hausdorff topological vector space E, Y be a convex subset of a topological vector space F, $S : X \rightrightarrows Y$ be a set-valued mapping and $f : E\times Y \to \mathbb{R}$ be a function. A Stampacchia-type variational inequality is stated as follows:

$(SVIP)$ Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in S(\bar{x})$ and $f(x-\bar{x}, \bar{y}) \geq 0$, for all $x \in X$.

Theorem 2.4. *Problem $SVIP$ has a solution if the set-valued mapping S is upper semicontinuous with nonempty, compact and convex values and the bifunction f satisfies the following conditions:*

- *f is continuous on $E \times Y$.*
- *For each $y \in Y$, $f(., y)$ is linear.*
- *For each $x \in E$, $f(x, .)$ is quasiconcave.*
- *For each $x \in X$ the set*

$$\{u \in X - X : \exists y \in S(x) \text{ such that } f(u, y) \geq 0\}$$

is convex.

In 2014, H. Huang and J. Zou proved the following result:

Theorem 2.5. *Problem (SVIP) has a solution if all conditions of theorem 2.2 are satiesfied but without condition (iv).*

We show that (SVIP) has a solution with less condition on f.

Theorem 2.6. *Problem $SVIP$ has a solution if the set-valued mapping S is upper semicontinuous with nonempty, compact and convex values and the bifunction f satisfies the following conditions:*

- *f is upper semicontinuous on $E \times Y$.*
- *For each $y \in Y$, $f(.,y)$ is convex and lower semicontinuous and for each $x \in E$, $f(x,.)$ is concave.*
- *For each $y \in Y$, $f(0,y) = 0$.*

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CLOSED RANGES OF TWO PRODUCT PROJECTIONS

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Abstract. In this paper, we study the closed range property and dense range property of modular operators on Hilbert C^*-modules. Also, some results about the idempotent product of projections are founded.

1. Introduction

The concept of generalized inverses of differential and integral operators antedated the generalized inverses of matrices, whose existence was first noted by E. H. Moore, who defined a unique inverse (called by him the general reciprocal) for every finite matrix (square or rectangular). In 1955 Penrose sharpened and extended Bjerhammars results on linear systems, and showed that Moores inverse, for a given matrix A,

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* Speaker.

is the unique matrix X satisfying the four equations (a)-(d) of Definition 1.5.The reader is referred to [1]–[7] and the references cited therein for more details.

Throughout this paper $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert $\mathcal{A}$-module is not necessarily orthogonally complemented, however Lance in [5] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1 (Theorem 3.2 in [5])**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range, and*

(i) $\ker(T)$ *is orthogonally complemented and* $(\ker(T))^{\perp} = \mathrm{ran}(T^*)$.
(ii) $\mathrm{ran}(T)$ *is orthogonally complemented in* $\mathcal{Y}$, *with* $(\mathrm{ran}(T))^{\perp} = \ker(T^*)$.

Definition 1.2. An operator T in $L(\mathcal{X}, \mathcal{Y})$ is said to be regular if there is S in $L(\mathcal{Y}, \mathcal{X})$ such that $TST = T$ and $STS = S$. In this case the operator S is called pseudo-inverse of T.

We can see that, the pseudo-inverse of a closed range operator T is unique. It is known that a bounded adjointable operator T has pseudo-inverse if and only if ran(T) is closed [8].

Note that $\mathcal{X}\mathcal{A}$ which is defined to be the linear span of $\{xa \mid x \in \mathcal{X}, a \in \mathcal{A}\}$ is dense in $\mathcal{X}$ and if $\mathcal{A}$ is unital, then $x.1 = x$. Clearly $\langle \mathcal{X}, \mathcal{X} \rangle = \mathrm{span}\{\langle x, y\rangle | x, y \in \mathcal{X}\}$ is a $*$-bi-ideal of $\mathcal{A}$.

Definition 1.3. If $\langle \mathcal{X}, \mathcal{X} \rangle$ is dense in $\mathcal{A}$, then $\mathcal{X}$ is called full. For example $\mathcal{A}$ as an $\mathcal{A}$-module is full.

Definition 1.4. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules. For any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we define $\theta_{x,y} : \mathcal{Y} \to \mathcal{X}$ by

$$\theta_{x,y}(z) = x\langle y, z\rangle \qquad (z \in \mathcal{Y}).$$

It is clear that $\theta_{x,y} \in L(\mathcal{Y}, \mathcal{X})$.

Definition 1.5. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^{\dagger}$ of T is an element in $L(\mathcal{Y}, \mathcal{X})$ which satisfies

(a) $TT^{\dagger}T = T$.
(b) $T^{\dagger}TT^{\dagger} = T^{\dagger}$.
(c) $(TT^{\dagger})^* = TT^{\dagger}$.
(d) $(T^{\dagger}T)^* = T^{\dagger}T$.

Motivated by these conditions, $T^{\dagger}$ is unique and $T^{\dagger}T$ and $TT^{\dagger}$ are orthogonal projections, in the sense that those are selfadjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^{\dagger} = (T^{\dagger})^*$. The following theorem is known.

Theorem 1.6 (Theorem 2.2 in [9])**.** *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.*

Remark 1.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T \in L(\mathcal{X}, \mathcal{Y})$ has closed range. Then $\mathrm{ran(TT^*)} = \mathrm{ran(T^*T)}$. Moreovere,

(i) $(TT^\dagger)^* = (TT^\dagger)$.
(ii) $(T^\dagger)^* T^* = TT^\dagger TT^\dagger$.
(iii) $(T^*)^\dagger T^* = TT^*(T^\dagger)^* T^\dagger$.
(iv) $(T^*)^\dagger T^\dagger TT^* = TT^*(TT^\dagger)^*$.
(v) $(TT^*)^\dagger TT^* = TT^*(TT^\dagger)^*$.

2. Main results

In this section, we give some results about dense range property and so some idempotent product of projections are presented.

Theorem 2.1. *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$–module such that any element $\mathcal{A}$ is invertible. Then for all x, y in $\mathcal{X}$, $\theta_{x,y}$ has pseudo-inverse operator.*

Corollary 2.2. *Let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$–module such that any element $\mathcal{A}$ is invertible. Then for all x_1, y_1, x_2, y_2 in $\mathcal{X}$, $\theta_{x_1,y_1}\theta_{x_2,y_2}$ has pseudo-inverse operator.*

At the sequal we discuss about dense property of the range of some modular operatoes. At following, we give the simple example of the dense range operator which is not closed range.

Example 2.3. Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $\{\lambda_n\}$ be a sequence of non-zero scalars such that $\lambda_n \to 0$ as $n \to 0$. Let $T \in L(\mathcal{Y}, \mathcal{X})$. For x in $\mathcal{X}$ and y in $\mathcal{Y}$, define

$$Tz = \sum_{j=1}^{\infty} \lambda_j \theta_{x,y}(z).$$

Note that T is a bounded linear operator, each λ_j is an eigenvalue of T and 0 is an accumulation point of the eigen spectrum of T. Consequently, $\mathrm{ran}(T)$ is not closed in $\mathcal{X}$, we observe that ran(T) is dense.

Theorem 2.4. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Hilbert $\mathcal{A}$-modules. Let $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and TS has closed range and T is an isometry with complemented range then S has closed range.*

Proposition 2.5. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Hilbert $\mathcal{A}$-modules. Let $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $(TS)^*$ has dense range and T is an isometry with complemented range then S^* has a dense range.*

An operator $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be unitary if $U^*U = 1_{\mathcal{X}}$ and $UU^* = 1_{\mathcal{Y}}$. If there exists a unitary element of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ then we say that $\mathcal{X}$ and $\mathcal{Y}$ are unitarily equivalent Hilbert $\mathcal{A}$-modules, and we write $\mathcal{X} \approx \mathcal{Y}$.

Theorem 2.6. *Suppose $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and there is an element $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that T and T^* have dense range and S has closed range. Then there exists an isometric $\mathcal{A}$-linear map $\mathcal{V}$ such that $\mathcal{V}S$ has closed range.*

Theorem 2.7. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has closed range and $T = P_{\mathrm{ran(T)}} P_{(\ker(T))^\perp}$ then the following statements hold.*

(i) $1 - T^\dagger T - TT^\dagger$ *has closed range.*
(ii) $\ker(T^*) + \mathrm{ran}(\mathrm{P}_{\mathrm{ran(T^*)}})$ *is an orthogonal complemented.*

Theorem 2.8. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $T \in L(\mathcal{X})$ has closed range and $T^\dagger$ be an idempotent operator. Then* $\ker(T^*) = \ker(T^*) \oplus (\mathrm{ran(T)} \cap \ker(\mathrm{T}))$.

Theorem 2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $T \in L(\mathcal{X}, \mathcal{Y})$ and $\mathcal{N}, \mathcal{M}$ are closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $P_{\mathcal{M}} T P_{\mathcal{N}}$ has closed range, then $T(P_{\mathcal{M}} T P_{\mathcal{N}})^\dagger$ is an idempotent closed range operator.*

Corollary 2.10. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{X}$ such that $P_{\mathcal{M}} P_{\mathcal{N}}$ has closed range, then $(P_{\mathcal{M}} P_{\mathcal{N}})^\dagger$ is an idempotent operator.*

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A CHARACTERIZATION OF EP OPERATORS ON HILBERT C^*-MODULES

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Abstract. In this paper, we present special reverse order law for the modular operators. Furthermore, we find some characterization of the EP operators in Hilbert C^*-modules.

1. Introduction

Hilbert C^*-modules form a category between Banach spaces and Hilbert spaces. The basic idea was to consider module over C^*-algebra instead of linear space and to allow the inner product to take values in a more general C^*-algebra than $\mathbb{C}$. The structure was first used by Kaplansky [3] in 1952 and more carefully investigated by Rieffel [9] and Paschke [8] later in 1972-73. We give only a brief introduction to the theory of Hilbert C^*-modules to make our explanations self-contained. For comprehensive accounts we refer to the lecture note of Lance [4].

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* Speaker.

On the other hand the concept of general inverese and in particular Moore-Penrose inverse is very important. The general inverese do not have all properties of the usual inverse. The reader is referred to [1] - [6] and the references cited therein for more details.

In this paper we specialize the investigations to the Moore-Penrose inverse of closed range operators on Hilbert C^*-modules. Throughout this paper we assume that $\mathcal{A}$ is an arbitrary C*-algebra and suppose $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. We use $[\cdot,\cdot]$ for commutator of two elements. The notations $\mathrm{Ker}(\cdot)$ and $\mathrm{Ran}(\cdot)$ stand for kernel and range of operators, respectively. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denotes the set of all bounded adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$, that is, all operator $T:\mathcal{X}\to\mathcal{Y}$ for which there exists $T^*:\mathcal{Y}\to\mathcal{X}$ such that $\langle Tx,y\rangle=\langle x,T^*y\rangle$, for all $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance in [4] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1 (Theorem 3.2 in [4])**.** *Suppose that $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then $T^*\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ has closed range, and*

(i) $\ker(T)$ *is orthogonally complemented and* $(\ker(T))^\perp=\mathrm{ran(T^*)}$.
(ii) $\mathrm{ran(T)}$ *is orthogonally complemented in $\mathcal{Y}$, with* $(\mathrm{ran(T)})^\perp=\ker(\mathrm{T}^*)$.

A generalized inverse of $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ is an operator $T^\times\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ such that

$$T\,T^\times T=T \qquad \text{and} \qquad T^\times T\,T^\times=T^\times. \tag{1.1}$$

It is known that a bounded adjointable operator T has generalized inverse if and only if $\mathrm{ran(T)}$ is closed.

Definition 1.2. Let $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$. The Moore-Penrose inverse of T (if it exists) is an element $T^\dagger$ of $\mathcal{L}(\mathcal{Y},\mathcal{X})$ satisfying

$$TT^\dagger T=T,\quad T^\dagger TT^\dagger=T^\dagger,\quad (TT^\dagger)^*=TT^\dagger,\quad (T^\dagger T)^*=T^\dagger T. \tag{1.2}$$

Motivated by these conditions, $T^\dagger$ is unique and $T^\dagger T$ and $T\,T^\dagger$ are orthogonal projections. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger=(T^\dagger)^*$. It is well-known that for invertible operators T and S(or nonsingular matrices) TS is invertible and $(TS)^{-1}=S^{-1}T^{-1}$. However, this so-called reverse order law is not necessarily true for other kind of generalized inverses. At the sequal, we discuss about the conditions provided that the reverse order law is hold.

2. Main Results

In this section, by using some block operator matrix techniques, we present sufficient and necessary conditions for EP operators.

Proposition 2.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed range, and let $S \in \mathcal{L}(\mathcal{Y})$ be self adjoint and invertible. Then* $\mathrm{ran(ST)} = \mathrm{ran(T)}$ *if and only if* $[S, TT^\dagger] = 0$.

Theorem 2.2. *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that S and T^* are surjective, then the reverse order law hold, i.e., $(TS)^\dagger = S^\dagger T^\dagger$. Moreover, if $U = TS$ then $UU^\dagger = TT^\dagger$, $U^\dagger U = S^\dagger S$, $T^\dagger = SU^\dagger$ and $S^\dagger = U^\dagger T$.*

Definition 2.3. Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. An operator $T \in \mathcal{L}(\mathcal{X})$ is called EP if $\mathrm{ran(T)}$ and $\mathrm{ran(T^*)}$ have the same closure.

In the Hilbert C^*-module context, one needs to add the extra condition, closeness of the range, in order to get a reasonably good theory. This ensures that an EP operator has a bounded adjointable Moore-Penrose inverse. Like the general theory of Hilbert spaces one can easily see that the following conditions are equivalent;

(i) T is EP with closed range.
(ii) T and T^* have the same kernel.
(iii) T is Moore-Penrose invertible and $TT^\dagger = T^\dagger T$.
(iv) $\mathrm{ran(T)}$ is orthogonally complemented in $\mathcal{X}$, with complement $\ker(T)$.

Theorem 2.4. *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that T and S^* are surjective. If $U = ST$, then the following statements are equivalent:*

(i) *U is EP.*
(ii) *$SS^\dagger = T^\dagger T$.*
(iii) *$S(S^*S)^\dagger S^* = S^*(TT^*)^\dagger T$.*
(iv) $\mathrm{ker(S^*)} = \mathrm{ker(T)}$.

Theorem 2.5. *Let $\mathcal{X}$ be Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X})$. Then the following conditions are equivalent:*

(i) *T is EP.*
(ii) *There exists a unique projection P such that $T + P$ is invertible and $TP = PT = 0$.*
(iii) *There exists a unitary operator $U \in \mathcal{L}(\mathcal{X})$ and an invertible operator X such that $T = U \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} U^*$.*

Theorem 2.6. *Suppose $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module, $T \in \mathcal{L}(\mathcal{X})$ has closed range and $S \in \mathcal{L}(\mathcal{X})$ is an arbitrary operator which commutes with T. Then $(1 - TT^{\dagger})S = S(1 - TT^{\dagger})$.*

Theorem 2.7. *Let $T \in \mathcal{L}(\mathcal{X})$ has closed range and $P \in \mathcal{L}(\mathcal{X})$ be a projection commuting with T. Then TP has closed range. Moreover, if T is an EP operator then $K = 1 - TPT^{\dagger} + TPT^{*}$ is invertible.*

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THE KAWAHARA EQUATION IN WEIGHTED SOBOLEV SPACES ON INFINITE INTERVALS

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Abstract. The initial-and boundary-value problem for the Kawahara equation, a fifth-order KdV type equation, is studied in weighted Sobolev space $L^2(e^x dx)$. The theory presented here includes the existence and uniqueness of a local mild solution of the Kawahara equation in a unbounded interval, under the effect of a localized damping mechanism.

1. Introduction

The Kawahara equation [1, 4]

$$\begin{aligned} &u_t + \beta u_{xxx} - u_{xxxxx} + uu_x + a(x)u = 0, \\ &u(0,t) = u_x(0,t) = 0, \qquad t > 0, \\ &u(x,0) = u_0(x), \qquad x > 0. \end{aligned} \tag{1.1}$$

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[*] Speaker.

is a dispersive PDE describing numerous wave phenomena as magento-acoustic in a cold plasma, the propagation of long waves in a shallow liquid beneath an ice sheet, gravity waves on the surface of a heavy liquid, etc. In the literature this equation is also referred as the fifth-order KdV equation, or singularly perturbed KdV equation [2]. Our aim here is to analyze qualitative properties of solutions to the initial-boundary value problem for (1.1) posed on infinite interval under the presence of a localized damping term. Here $\beta > 0$ and $a = a(x)$ is a non-negative function belonging to $L^\infty(0, +\infty)$ and moreover, in the most of the paper we will assume that $a(x) \geq a_0 > 0$ a.e. in an open, nonempty subset of $(0, +\infty)$, where the damping is acting effectively.

In this paper, following the work by Vasconcellos and Silva [3] on the Kawahara equation, we study the existence and uniqueness of local mild solutions for the Kawahara system in $C([0, T]; L^2(0, +\infty)) \cap L^2_{loc}(0, T; H^2(0, +\infty))$. Let $\Omega = (0, +\infty)$. Define

$$L^2_e(\Omega) = \left\{ u : \Omega \to R \,|\, \int_\Omega |u(x)|^2 e^x dx < \infty \right\}.$$

The following weighted Sobolev spaces

$$H^s_e(\Omega) = \left\{ u : \Omega \to R \,|\, \partial^i_x u \in L^2_e,\ 0 \leq i \leq s;\ u(0) = u_x(0) = 0 \ \textit{if}\ s \geq 1 \right\}.$$

endowed with their usual inner product, will be used thereafter. Note that $H^0_e = L^2_e$.

2. MILD SOLUTION

Let us consider the operator $A : D(A) \subset L^2_e \to L^2_e$ with domain

$$D(A) = \{ u \in H^5_e(\Omega) \,|\, u(0) = u_x(0) = 0 \}$$

defined by $Au = \beta u_{xxx} - u_{xxxxx} + a(x)u$. Then, the following result holds.

Lemma 2.1. *Let $\beta > 10/12$. Then, the operator $-A$ defined above generates a continuous semigroup of operators $\{S(t)\}_{t \geq 0}$ in L^2_e.*

Proof. We first introduce the new variable $v = e^{\frac{x}{2}} u$ and consider the following initial-and boundary-value problem

$$\begin{aligned} & v_t + \beta(\partial_x - \frac{1}{2})^3 v - (\partial_x - \frac{1}{2})^5 v + a(x) v = 0, \\ & v(0, t) = v_x(0, t) = 0, \qquad t > 0, \\ & v(x, 0) = v_0(x) = e^{\frac{x}{2}} u_0(x), \qquad x > 0. \end{aligned} \tag{2.1}$$

Clearly, the operator $B : D(B) \subset L^2 \to L^2$ with domain

$$D(B) = \{ v \in H^5(\Omega) \,|\, v(0) = v_x(0) = 0 \}$$

defined by

$$Bv = -\beta(\partial_x - \frac{1}{2})^3 v + (\partial_x - \frac{1}{2})^5 v - a(x)v$$

is densely defined and closed. So, we are done if we prove that for some real number λ the operator $B - \lambda$ and its adjoint $B^* - \lambda$ are both dissipative in $L^2(\Omega)$. It is readily seen that B^* is given by $B^* v = \beta(\partial_x + \frac{1}{2})^3 v - (\partial_x + \frac{1}{2})^5 v + a(x)v$ with domain $D(B^*) = \{v \in H^5(\Omega) \,|\, v(0) = v_x(0) = v_{xx}(0) = 0\}$. Pick any $v \in D(B)$. After some integration by parts, we obtain that

$$\begin{aligned}(Bv, v)_{L^2} = &-\frac{1}{2}v_{xx}^2(0) + \frac{10-12\beta}{8}\int_\Omega v_x^2 dx + \frac{4\beta}{32}\int_\Omega v^2 dx \\ &-\frac{5}{2}\int_\Omega v_{xx}^2 dx - \int_\Omega a(x)v^2 dx - \frac{1}{32}\int_\Omega v^2 dx, \end{aligned} \tag{2.2}$$

that is $([B - \frac{4\beta}{32}]v, v)_{L^2} \leq 0$. Analogously, we deduce that for any $v \in D(B^*)$, $(v, [B^* - \frac{4\beta}{32}]v)_{L^2} \leq 0$ which completes the proof. □

Definition 2.2. A mild solution of the initial-and boundary-value problem (1.1) is a function $u \in F = C([0,T]; L_e^2(\Omega)) \bigcap L^2(0,T; H_e^2(\Omega))$ satisfying

$$\frac{du}{dt} + Au = -N(u,u), \tag{2.3}$$

where $N(u,v) = (uv)_x$.

3. Existence and Uniqueness

To prove the existence of a solution of (1.1) we introduce the map Ψ defined by

$$(\Psi u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u,u)(s)ds. \tag{3.1}$$

Theorem 3.1. *Let $5/3 < \beta < 2a_0 + 1$ and $u_0 \in L_e^2$. Then there exists $T = T(\|u_0\|_{L_e^2})$ such that the initial-and boundary-value problem (1.1) has a unique mild solution u satisfying $u \in F$. In addition, u obeys the bound*

$$\begin{aligned}\|u(t)\|_{L_e^2}^2 + \gamma\int_0^t \|u(\tau)\|_{H_e^2}^2 d\tau \leq \|u_0\|_{L_e^2}^2 \\ + \gamma\int_0^t \|u(\tau)\|_{L_e^2}^3 \|u(\tau)\|_{H_e^2} d\tau, \end{aligned} \tag{3.2}$$

where $\gamma > 0$ is constant.

Proof. To prove this theorem, one applies the contraction mapping principle to the integral equation $u(t) = (\Psi u)(t)$. Let $R = \|u_0\|_{L^2_e}$ and $B_{2R} = \{u \in F \mid \|u\|_F \leq 2R\}$ where

$$\|u\|_F = \sup_{t\in[0,T]} \|u(t)\|_{L^2_e} + \|u\|_{L^2(0,T;H^2_e(\Omega))}.$$

One shows that the map $\Psi(u)$ defined by (3.1), defines a contraction mapping from B_{2R} to B_{2R}. Let $u \in B_{2R}$. $\Psi(u)$ satisfies in (2.3) and one obtains after taking the inner product of this equation with $\Psi(u)$,

$$\frac{d}{dt}\|\Psi(u)\|^2_{L^2_e} + 2(A\Psi(u), \Psi(u))_{L^2_e} = -2(N(u,u), \Psi(u))_{L^2_e}. \tag{3.3}$$

Now, if we choose $5/3 < \beta < 2a_0 + 1$, then by relation $\int_\Omega u_x^2 e^x dx < 4\int_\Omega u_{xx}^2 e^x dx$ and integrating by parts, one has

$$(Au, u)_{L^2_e} = \int_\Omega (\beta u_{xxx} - u_{xxxxx} + a(x)u)ue^x dx \geq \gamma\|u\|^2_{H^2_e}, \tag{3.4}$$

where $\gamma > 0$.

On the other hand, for any $(u, v) \in H^2_e \times H^2_e$, we have

$$\|N(u,v)\|_{(H^2_e)'} \leq C\|u\|_{L^2_e}\|v\|^{\frac{1}{2}}_{L^2_e}\|v\|^{\frac{1}{2}}_{H^2_e}$$

and thus we deduce that

$$2|(N(u,u), \Psi(u))_{H^2_e}| \leq \gamma\|\Psi(u)\|^2_{H^2_e} + C\|u\|^3_{L^2_e}\|u\|_{H^2_e}. \tag{3.5}$$

Therefore, combining (3.3), (3.4) and (3.5) yields (3.2). If we choose $T > 0$ such that $R^2 + 16C\sqrt{T}R^4 < 2\min\{1,\gamma\}R^2$, then $\|\Psi(u)\|_F < 2R$.

To show Ψ is a contraction, first note that

$$\Psi(u) - \Psi(v) = -\int_0^T S(t-s)(N(u-v,u) + N(v,u-v))ds.$$

A similar process as in the estimation of $\|\Psi(u)\|_{H^2_e}$ yields

$$\|\Psi(u) - \Psi(v)\|^2_{L^2_e} + \gamma\int_0^T \|\Psi(u) - \Psi(v)\|^2_{H^2_e} ds \leq C\sqrt{T}\|u - v\|^2_F(\|u\|^2_F + \|v\|^2_F).$$

If T is further restricted to $4C\sqrt{T}R^2 < \min(1,\gamma)$, then

$$\|\Psi(u) - \Psi(v)\|_F \leq \vartheta\|u - v\|_F,$$

where $\vartheta^2 = \frac{4C\sqrt{T}R^2}{\min(1,\gamma)} < 1$. Applying the contraction mapping principle, the proof will be complete. $\square$

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STRICT INNER AMENABILITY FOR TENSOR PRODUCT OF HOPF-VON NEUMANN ALGEBRAS

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Abstract. In this paper for two Hopf-von Neumann algebras $\mathbb{H}_1 = (\mathfrak{M}_1, \Gamma_1)$ and $\mathbb{H}_2 = (\mathfrak{M}_2, \Gamma_2)$, we prove that if $\mathbb{H}_1$ is strictly inner amenable and either $\mathbb{H}_2$ is strictly inner amenable or predual of $\mathfrak{M}_2$ has a bounded approximate identity, then tensor product of $\mathbb{H}_1$ and $\mathbb{H}_2$ is strictly inner amenable.

1. Introduction and Preliminaries

Throughout this paper, the continuous dual of a normed space $\mathfrak{X}$ is denoted by $\mathfrak{X}^*$ and the value of $\phi \in \mathfrak{X}^*$ at $\xi \in \mathfrak{X}$ is denoted by $\phi(\xi)$ or $\langle \phi, \xi \rangle$. Also, if $\mathfrak{X}$ has a predual, then the predual of $\mathfrak{X}$ is denoted by $\mathfrak{X}_*$. Moreover, we use $\mathfrak{M}$ to denote a von-Neumann algebra with identity element 1 and $\bar{\otimes}$ to denote the von Neumann algebra tensor product.

Recall that a pair $\mathbb{H} = (\mathfrak{M}, \Gamma)$ is a *Hopf-von Neumann algebra* when $\Gamma : \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ is a co-multiplication; i.e., a normal, unital, and $*$-homomorphism satisfying

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma,$$

where id is the identity map form $\mathfrak{M}$ into itself.

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For a Hopf-von Neumann algebra $\mathbb{H} = (\mathfrak{M}, \Gamma)$, one can define a product $*$ on $\mathfrak{M}_*$, turning it into Banach algebra, by

$$\langle f * g, x\rangle = \langle f \otimes g, \Gamma x\rangle$$

for all $f, g \in \mathfrak{M}_*$ and $x \in \mathfrak{M}$.

Lau [1] has been introduced and studied a large class of Banach algebras, called F-algebras, a complex Banach algebra $\mathcal{L}$ which is the (unique) predual of a von Neumann algebra $\mathfrak{M}$ whose identity element 1 is a multiplicative linear functional on $\mathcal{L}$; Later on, F-algebras were termed Lau algebras by Pier [3]. Therefore, it is easy to see that for a Hopf-von Neumann algebra $\mathbb{H} = (\mathfrak{M}, \Gamma)$ the unique predual $\mathfrak{M}_*$ is a Lau algebra. Nasr-Isfahani [4] introduced and studied inner and strict inner amenability of Lau algebras. In particular, we say that a Hopf-von Neumann algebra $(\mathfrak{M}, \Gamma)$ is *inner amenable* if there exists an *inner invariant mean* on $\mathfrak{M}$; i.e., there exists a functional $M \in \mathfrak{M}^*$ such that $\|M\| = M(1) = 1$ and $M(f \cdot x) = M(x \cdot f)$, or equivalently $f \odot M = M \odot f$ for all $x \in \mathfrak{M}$ and $f \in \mathfrak{M}_*$, where the first Arens multiplication $\odot$ is defined by the equations

$$\langle F \odot H, x\rangle = \langle F, Hx\rangle, \quad \langle Hx, f\rangle = \langle H, x \cdot f\rangle, \quad \langle x \cdot f, g\rangle = \langle x, f * g\rangle$$

for all $F, H \in \mathfrak{M}^*$, $x \in \mathfrak{M}$, and $f, g \in \mathfrak{M}_*$. Moreover, $\langle f \cdot x, g\rangle = \langle x, g * f\rangle$.

2. STRICT INNER AMENABILITY

Let $(\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra and recall that an element $E \in \mathfrak{M}^*$ is called a *mixed identity* if $f \odot E = E \odot f = f$ or equivalently, $E(x \cdot f) = E(f \cdot x)$ for all $f \in \mathfrak{M}_*$ and $x \in \mathfrak{M}$. It is known that $E \in \mathfrak{M}^*$ is a mixed identity if and only if it is a weak* cluster point of a bounded approximate identity in $\mathfrak{M}_*$.

Since all Hopf-von Neumann algebras with a bounded approximate identity in $\mathfrak{M}_*$ are always inner amenable, this leads us to the following definition.

Definition 2.1. Let $\mathbb{H} = (\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra with a bounded approximate identity in $\mathfrak{M}_*$. We say that $\mathbb{H}$ is *stictly inner amenable* if there exists an inner invariant mean in $\mathfrak{M}^*$ such that it is not a mixed identity.

Note that for a Hopf-von Neumann algebra $\mathbb{H} = (\mathfrak{M}, \Gamma)$, we use $P_1(\mathbb{H})$ to denote the set of all positive functionals in $\mathfrak{M}_*$ with norm one; i.e.,

$$P_1(\mathbb{H}) = \{f \in \mathfrak{M}_* \ : \ \|f\| = \langle f, 1\rangle = 1\}.$$

Theorem 2.2. *Let $\mathbb{H} = (\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra. Then $\mathbb{H}$ is strictly inner amenable if and only if there is a net (f_α) in $P_1(\mathbb{H})$ such that $\|f * f_\alpha - f_\alpha * f\| \to 0$ for all $f \in \mathfrak{M}_*$ and any weak* cluster point (f_α) in $\mathfrak{M}^*$ is not a mixed identity.*

Proof. Suppose that $\mathbb{H}$ is strictly inner amenable and choose a strict inner invariant mean $F \in \mathfrak{M}^*$. By Goldestine's Theorem, there exists a bounded net $(f_\beta) \subseteq \mathfrak{M}_*$ such that $f_\beta \to F$ in the weak* topology of $\mathfrak{M}^*$. Since $f_\beta(1) \to \langle F, 1\rangle = 1$, without loss of generality, we can assume $f_\beta(1) = 1$ for all β and it is clear that $f * f_\beta - f_\beta * f \to 0$ in the weak topology of $\mathfrak{M}_*$ for all $f \in \mathfrak{M}_*$. A standard argument similar to the proof of Theorem 1, Page 524 in [2], shows that we can find a net (f_α) consisting of convex combinations of elements in (f_β) such that $\|f * f_\alpha - f_\alpha * f\| \to 0$ for all $f \in \mathfrak{M}_*$ and $f_\alpha \to F$ in the weak* topology of $L^\infty(\mathbb{G})^*$. The converse is trivial. ☐

3. Tensor Product and Strictly Inner Amenability

Let $\mathbb{H}_1 = (\mathfrak{M}_1, \Gamma_1)$ and $\mathbb{H}_2 = (\mathfrak{M}_2, \Gamma_2)$ be two Hopf-von Neumann algebras. Define $\mathfrak{M} := \mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2$, and

$$\Gamma = \Gamma_1 \times \Gamma_2 := (\mathrm{id}_{\mathfrak{M}_1} \otimes \sigma \otimes \mathrm{id}_{\mathfrak{M}_2})(\Gamma_1 \otimes \Gamma_2)$$

where σ is the flip mapping $\sigma(x \otimes y) = y \otimes x$ from $\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2$ into $\mathfrak{M}_2 \bar{\otimes} \mathfrak{M}_1$. Then it is not difficult to see that Γ is a co-multiplication of $\mathfrak{M}$. Therefore $(\mathfrak{M}, \Gamma)$ is a Hopf-von Neumann algebra that we denote it by $\mathbb{H}_1 \otimes \mathbb{H}_2$; i.e.,

$$\mathbb{H}_1 \otimes \mathbb{H}_2 = (\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2, (\mathrm{id}_{\mathfrak{M}_1} \otimes \sigma \otimes \mathrm{id}_{\mathfrak{M}_2})(\Gamma_1 \otimes \Gamma_2)).$$

It is known that the predual of $\mathfrak{M}_1 \bar{\otimes} \mathfrak{M}_2$ is $\mathfrak{M}_{1*} \widehat{\otimes} \mathfrak{M}_{2*}$, where $\widehat{\otimes}$ is the operator space projective tensor product and the norm $\|.\|_\wedge$ on $\mathfrak{M}_{1*} \widehat{\otimes} \mathfrak{M}_{2*}$ is a subcross matrix norm, i.e., $\|f \otimes g\|_\wedge \le \|f\|\|g\|$ for all $f \in \mathfrak{M}_{1*}$ and $g \in \mathfrak{M}_{2*}$. Now we have the main theorem of this paper.

Theorem 3.1. *Let $\mathbb{H}_1 = (\mathfrak{M}_1, \Gamma_1)$ and $\mathbb{H}_2 = (\mathfrak{M}_2, \Gamma_2)$ be two Hopf-von Neumann algebras. If $\mathbb{H}_1$ is strictly inner amenable and either $\mathbb{H}_2$ is strictly inner amenable or $\mathbb{H}_2$ has a bounded approximate identity in $\mathfrak{M}_{2*}$, then $\mathbb{H}_1 \otimes \mathbb{H}_2$ is strictly inner amenable.*

Proof. First, suppose that $\mathbb{H}_1 = (\mathfrak{M}_1, \Gamma_1)$ and $\mathbb{H}_2 = (\mathfrak{M}_2, \Gamma_2)$ are strictly inner amenable. By Theorem 2.2, there exist bounded nets $(f_\alpha) \subseteq P_1(\mathbb{H}_1)$ and $(g_\beta) \subseteq P_1(\mathbb{H}_2)$ such that $\|f * f_\alpha - f_\alpha * f\| \to 0$ and $\|g * g_\beta - g_\beta * g\| \to 0$ for all $f \in \mathfrak{M}_{1*}$ and $g \in \mathfrak{M}_{2*}$.

Without loss of generality, we can assume that $f_\alpha \to F$ in the weak* topology of $\mathfrak{M}_1^*$ such that $F \in \mathfrak{M}_1^*$ is not a mixed identity. Set

$$w_{(\alpha,\beta)} := f_\alpha \otimes g_\beta$$

for all α and β. Consider the bounded net $(w_{(\alpha,\beta)}) \subseteq P_1(\mathbb{H}_1 \otimes \mathbb{H}_2)$. Thus for every $f \in \mathfrak{M}_{1*}$ and $g \in \mathfrak{M}_{2*}$ we have

$$\|(f \otimes g)w_{(\alpha,\beta)} - w_{(\alpha,\beta)}(f \otimes g)\|_\wedge = \|f * f_\alpha \otimes g * g_\beta - f_\alpha * f \otimes g_\beta * g\|_\wedge = I.$$

Therefore

$$\begin{aligned} I &\leq \|f * f_\alpha \otimes g * g_\beta - f_\alpha * f \otimes g * g_\beta\|_\wedge + \\ &\qquad \|f_\alpha * f \otimes g * g_\beta - f_\alpha * f \otimes g_\beta * g\|_\wedge \\ &\leq \|f * f_\alpha - f_\alpha * f\|\|g * g_\beta\| + \|f_\alpha * f\|\|g * g_\beta - g_\beta * g\| \to 0. \end{aligned}$$

Thus, it is easy to see that $\|w * w_{(\alpha,\beta)} - w_{(\alpha,\beta)} * w\|_\wedge \to 0$ for all $w \in \mathfrak{M}_{1*}\widehat{\otimes}\mathfrak{M}_{2*}$. Now, we claim that any weak* cluster point of $w_{(\alpha,\beta)}$ is not a mixed identity. To prove this, let H be a weak* cluster point of $w_{(\alpha,\beta)}$ in $(\mathfrak{M}_1\bar{\otimes}\mathfrak{M}_2)^*$. Then there is a subnet $(w_{(\gamma,\delta)})$ of $(w_{(\alpha,\beta)})$ convergent to H in the weak* topology of $(\mathfrak{M}_1\bar{\otimes}\mathfrak{M}_2)^*$. Consider the map $\Psi : \mathfrak{M}_{1*}\widehat{\otimes}\mathfrak{M}_{2*} \to \mathfrak{M}_{1*}$ defined by

$$\Psi(f \otimes g) = \langle g, 1\rangle f$$

for all $f \in \mathfrak{M}_{1*}$ and $g \in \mathfrak{M}_{2*}$. It is easy to check that Ψ is a continuous epimorphism when $\mathfrak{M}_{1*}\widehat{\otimes}\mathfrak{M}_{2*}$ and $\mathfrak{M}_{1*}$ have the weak topology.

Now, if H is a mixed identity in $(\mathfrak{M}_1\bar{\otimes}\mathfrak{M}_2)^*$, then $w * w_{(\gamma,\delta)} - w \to 0$ in the weak topology of $\mathfrak{M}_{1*}\widehat{\otimes}\mathfrak{M}_{2*}$ for all $w \in \mathfrak{M}_{1*}\widehat{\otimes}\mathfrak{M}_{2*}$. Consequently,

$$\Psi(w) * \Psi(w_{(\gamma,\delta)}) - \Psi(w) = \Psi(w * w_{(\gamma,\delta)} - w) \to 0$$

in the weak topology of $\mathfrak{M}_{1*}$. It follows that $f * f_\gamma g_\delta(1) - f \to 0$ in the weak topology of $\mathfrak{M}_{1*}$ for all $f \in \mathfrak{M}_{1*}$. On the other hand, $f_\gamma g_\delta(1) \to F$ in the weak* topology of ${\mathfrak{M}_1}^*$. These show that $f \odot F = f$ and similarly $F \odot f = f$ for all $f \in \mathfrak{M}_{1*}$; that is, F is a mixed identity in ${\mathfrak{M}_1}^*$ which is a contradiction.

A similar way shows that if $\mathbb{H}_1$ is strictly inner amenable and $\mathbb{H}_2$ has a bounded approximate identity in $\mathfrak{M}_{2*}$, then $\mathbb{H}_1 \otimes \mathbb{H}_2$ is strictly inner amenable. $\square$

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AN INTEGRAL BOUND FOR p-ANGULAR DISTANCE

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Abstract. In this paper, we obtain integral bounds involving two different p-angular distances, which generalizes some results due to Dragomir from [S. S. Dragomir, *New inequalities for the p-angular distance in normed spaces with applications*, Ukrainian Math. J., 67 (2015), 19-32.].

1. Introduction

The angular distance between non-zero elements x and y in a normed linear space $\mathcal{X}$ over the field of real numbers is define by

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

In [4], Maligranda considered the p-angular distance

$$\alpha_p[x,y] = \big\| \|x\|^{p-1}x - \|y\|^{p-1}y \big\| \qquad (p \in \mathbb{R})$$

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* Speaker.

between non-zero vectors x and y in $\mathcal{X}$ as a generalization of the concept of angular distance.

Dunkl and Williams obtained a useful upper bound for the angular distance. They showed that for each nonzero vectors $x, y \in \mathcal{X}$,

$$\alpha[x,y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}.$$

Also, we recall the result of Hile [3]:

$$\alpha_p[x,y] \leq \frac{\|y\|^p - \|x\|^p}{\|y\|-\|x\|}\|x-y\|, \tag{1.1}$$

for $p \geq 1$ and $x, y \in \mathcal{X}$ with $\|x\| \neq \|y\|$.

For some recently obtained upper and lower bounds for the p-angular distance which mainly are comparison of α_p with α_1, the reader is referred to [1, 2, 4, 5] and references therein.

In [2], Dragomir showed that if $p \geq 1$, then for any $x, y \in \mathcal{X}$,

$$\alpha_p[x,y] \leq p\ \|x-y\| \int_0^1 \|(1-t)x+ty\|^{p-1}\,dt. \tag{1.2}$$

In this paper, we give some results comparing two different p-angular distances with each other, which generalize some results due to Dragomir [2]. The advantage of taking p and q arbitrary is that, whenever we find an inequality involving α_p and α_q, we can obtain its reverse by changing the roles of p and q with each other, which is as sharp as the first one.

2. Integral Bound

The following theorem yields the result of Dragomir in [2], if we take $q = 1$.

Theorem 2.1. *Let $x, y \in \mathcal{X} \setminus \{0\}$, $p, q \in \mathbb{R}$ and $q \neq 0$.*

(i) *If $p/q \geq 1$, then*

$$\alpha_p[x,y] \leq \frac{p}{q}\ \alpha_q[x,y] \int_0^1 \left\|(1-t)\|x\|^{q-1}x+t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1}dt. \tag{2.1}$$

(ii) *If $p/q < 1$ and x, y are linearly independent, then*

$$\alpha_p[x,y] \leq \frac{2q-p}{q}\ \alpha_q[x,y] \int_0^1 \left\|(1-t)\|x\|^{q-1}x+t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1}dt. \tag{2.2}$$

Corollary 2.2. *Let $x, y \in \mathcal{X}$ be linearly independent and $p, q \in \mathbb{R} \setminus \{0\}$.*

(i) *If $0 < p/q \leq 1$, then*

$$\alpha_p[x,y] \geq \frac{p}{q}\alpha_q[x,y]\left(\int_0^1 \left\|(1-t)\|x\|^{p-1}x+t\|y\|^{p-1}y\right\|^{\frac{q}{p}-1}dt\right)^{-1}.$$

(ii) *If $p/q \geq 1$ or $p/q < 0$, then*

$$\alpha_p[x,y] \geq \frac{p}{2p-q}\alpha_q[x,y]\left(\int_0^1 \left\|(1-t)\|x\|^{p-1}x + t\|y\|^{p-1}y\right\|^{\frac{q}{p}-1}\,dt\right)^{-1}.$$

Remark 2.3. (i) If $p/q \geq 1$, then, by the triangle inequality, we have

$$\left\|(1-t)\|x\|^{q-1}x + t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1} \leq [(1-t)\|x\|^q + t\|y\|^q]^{\frac{p}{q}-1}$$

for any $t \in [0,1]$. Integrating both sides on $[0,1]$, we get

$$\int_0^1 \left\|(1-t)\|x\|^{q-1}x + t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1}\,dt \leq \frac{q}{p}\left(\frac{\|y\|^p - \|x\|^p}{\|y\|^q - \|x\|^q}\right)$$

if $\|x\| \neq \|y\|$, and by (2.1) we obtain the chain of inequalities

$$\begin{aligned}\alpha_p[x,y] &\leq \frac{p}{q}\,\alpha_q[x,y]\int_0^1 \left\|(1-t)\|x\|^{q-1}x + t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1}\,dt\\ &\leq \frac{\|y\|^p - \|x\|^p}{\|y\|^q - \|x\|^q}\alpha_q[x,y], \end{aligned} \tag{2.3}$$

which provides a generalization and refinement of Hile's inequality.

(ii) If $p/q \geq 2$, then the function $f : [0,1] \to [0,\infty)$ given by $f(t) = [(1-t)\|x\|^q + t\|y\|^q]^{\frac{p}{q}-1}$ is convex. Employing the Hermite-Hadamard inequality for the convex function f, we obtain

$$\begin{aligned}\frac{q}{p}\left(\frac{\|y\|^p - \|x\|^p}{\|y\|^q - \|x\|^q}\right) &= \int_0^1 [(1-t)\|x\|^q + t\|y\|^q]^{\frac{p}{q}-1}dt\\ &\leq \frac{\|x\|^{p-q} + \|y\|^{p-q}}{2} \leq \max\{\|x\|^{p-q}, \|y\|^{p-q}\},\end{aligned}$$

which by (2.1), implies for $\|x\| \neq \|y\|$ the following sequence of inequalities

$$\begin{aligned}\alpha_p[x,y] &\leq \frac{p}{q}\,\alpha_q[x,y]\int_0^1 \left\|(1-t)\|x\|^{q-1}x + t\|y\|^{q-1}y\right\|^{\frac{p}{q}-1}\,dt\\ &\leq \frac{p}{q}\,\alpha_q[x,y]\int_0^1 [(1-t)\|x\|^q + t\|y\|^q]^{\frac{p}{q}-1}dt\\ &= \frac{\|y\|^p - \|x\|^p}{\|y\|^q - \|x\|^q}\alpha_q[x,y]\\ &\leq \frac{p}{q}\alpha_q[x,y]\frac{\|x\|^{p-q} + \|y\|^{p-q}}{2} \leq \frac{p}{q}\alpha_q[x,y]\max\{\|x\|^{p-q}, \|y\|^{p-q}\}.\end{aligned}$$

Remark 2.4. Let $\mathcal{X}$ be an inner product space. It is known that for any $a, b \in \mathcal{X}$, $b \neq 0$, it holds that

$$\min_{t\in\mathbb{R}}\|a + tb\| = \frac{\sqrt{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}}{\|b\|}.$$

Hence, if x and y are linearly independent vectors of $\mathcal{X}$, then by taking $a = x$ and $b = y - x$, we obtain

$$\|(1-t)x + ty\| = \|x + t(y-x)\| \geq \frac{\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}}{\|x-y\|} \quad (t \in \mathbb{R}).$$

This implies that

$$\int_0^1 \|(1-t)x + ty\|^{-1} dt \leq \frac{\|x-y\|}{\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}}.$$

Taking $p = 0$ and $q = 1$ in (2.2), we get

$$\alpha[x, y] \leq 2\|x-y\| \int_0^1 \|(1-t)x + ty\|^{-1} dt \leq \frac{2\|x-y\|^2}{\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}}.$$

This implies an upper estimation for the error of the Cauchy-Schwarz inequality as follows

$$\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2} \leq \frac{2\|x\|\|y\|\|x-y\|^2}{\|\|y\|x - \|x\|y\|} \qquad (\|y\|x \neq \|x\|y).$$

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SOME CHARACTERIZATIONS OF SURJECTIVE OPERATORS ON BANACH LATTICES

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Abstract. In this paper, we will study some properties of b-weakly compact operators and its relationships with other properties of positive operators. We investigate conditions under which operators on Banach lattices must be surjective.

1. Introduction

An operator T from a Banach space X into a Banach space Y is compact (respectively, weakly compact) if $\overline{T(B_X)}$ is compact (respectively, weakly compact) where B_X is the closed unit ball of X. There exists compact operator T from a Banach space X into a Banach space Y which is not surjective. Indeed, the operator $T : \ell^1 \to \ell^\infty$ defined by $T(\alpha_n) = (\Sigma_{n=1}^{\infty}\alpha_n, \Sigma_{n=1}^{\infty}\alpha_n, \ldots)$ is a compact operator which is not surjective. An operator T from a Banach space X into a Banach space Y is weakly compact if X or Y is reflexive, but the converse is false in general. In fact, each continuous operator from ℓ^∞ into c_0 is weakly compact, but ℓ^∞ and c_0 are not reflexive. By Corollary 2.6 from [3], we see that if E is an infinite-dimensional AL-space and F a Banach

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lattice and $T : E \to F$ is an arbitrary weakly compact operator, then F is reflexive. Not that if an operator T from a Banach space X into a Banach space Y is compact (respectively, weakly compact) and surjective, then Y must be finite-dimensional (respectively, Y must be reflexive).

For terminology concerning Banach lattice theory and positive operators we refer the reader to [1].

2. Main results

Recall that if an operator T from a Banach space X into a Banach space Y is compact and $T(X)$ is closed, then $T(X)$ is finite-dimensional. So, if $T : X \to Y$ is a surjective compact operator between Banach spaces, then Y is finite-dimensional. Consequently, if X is an infinite-dimensional Banach space then there is no surjective compact operator on X. Let $T : X \to Y$ be a weakly compact operator between Banach spaces. If $T(X)$ is closed, then $T(X)$ is reflexive. So, if $T : X \to Y$ is a surjective weakly compact operator between Banach spaces, then Y is reflexive. An operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\epsilon > 0$, there exists some $u \in F^+$ such that, $T(B_E) \subset [-u, u] + \epsilon B_F$ where $F^+ = \{x \in F : x \geq 0\}$. Note that the identity operator $i : \ell^\infty \to \ell^\infty$ is a surjective semi-compact operator, but the operator $T : \ell^\infty \to \ell^\infty$ defined by $T(\alpha_n) = (\alpha_1, \alpha_2/2, \alpha_3/3, ...)$ is a semi-compact operator which is not surjective.

Theorem 2.1. *Let E and F be two Banach lattices such that F has an order continuous norm. If $T : E \to F$ is a positive surjective semi-compact operator, then F is reflexive.*

Corollary 2.2. *Let E be a non-reflexive Banach lattice with an order continuous norm. Then there is no positive surjective semi-compact operator on E.*

Recall that a nonzero element x of a Banach lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements. For example the Banach lattice ℓ^2 is discrete while $L^1[0, 1]$ is not.

Theorem 2.3. *Let E and F be Banach lattices. Suppose that $T : E \to F$ is a positive injective semi-compact operator such that its range is closed. If one of the following statements is valid, then F is reflexive.*

(a) *F is discrete and its norm is order continuous.*

(b) *E' is discrete of order continuous norm and F has an order continuous norm.*
(c) *The norms of E, E' and F are order continuous and E has the Dunford-pettis property (i.e., each weakly compact operator from E into an arbitrary Banach space F is Dunford-Pettis).*

An operator T from a Banach space E into another F is said to be Dunford-Pettis, if it carries each weakly compact subset of E onto a compact subset of F (i.e., whenever $x_n \xrightarrow{\omega} 0$ implies $Tx_n \xrightarrow{\|.\|} 0$). It is clear that any compact operator is Dunford-Pettis, while a Dunford-Pettis operator is not necessarily compact. Indeed, the identity operator $i : \ell^1 \to \ell^1$ is Dunnford-Pettis but is not compact. The operator $T : \ell^\infty \to \ell^\infty$ defined by $T(\alpha_n) = (\alpha_1\alpha_2/2, \alpha_3/3, ...)$ is a Dunford-Pettis operator which is not surjective.

Theorem 2.4. *Let E and F be Banach lattices and $T : E \to F$ be a positive surjective Dunford-Pettis operator. If one of the following conditions is valid, then F is reflexive.*

(a) *The norm of E and E' are order continuous.*
(b) *E' is discrete and its norm is order continuous.*
(c) *The norm of E' is order continuous, F is discrete and its norm is order continuous.*

Proof. Let E and F be Banach lattices and $T : E \to F$ be a positive Dunford-Pettis operator. By using Theorem 2.2 from [4], T is a compact operator. So, F is finite dimensional, and hence F is reflexive. $\square$

Theorem 2.5. *Let E and F be Banach lattices such that F is not reflexive. If a positive Dunford-Pettis operator $T : E \to F$ is surjective, then the norm of E' is not order continuous.*

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact, if it maps each subset of E which is b-order bounded (i.e., order bounded in the topological bidual E'') into a relatively weakly compact subset of X.

Theorem 2.6. *Let E and F be Banach lattices and $T : E \to F$ be a positive surjective b-weakly compact operator. Then the norm of F is order continuous.*

Recall that if E is a Banach lattice and if $0 \leqslant x'' \in E''$, then the principal ideal $I_{x''}$ generated by $x'' \in E''$ under the norm $\|.\|_\infty$ defined by

$$\|y''\|_\infty = \inf\{\lambda > 0 : \ |y''| \leq \lambda x''\}, \qquad (y^{''} \in I_{x''})$$

is an AM-space with unit x'', whose closed unit ball is order interval $[-x'', x'']$.

Lemma 2.7. *Let E be a Banach lattice. Then every b-order bounded disjoint sequence in E is weakly convergent to zero.*

Theorem 2.8. *Every Dunford-Pettis operator from a Banach lattice E into a Banach space X is b-weakly compact.*

Recall that if E is a Banach lattice then E^a is the maximal order ideal in E on which the norm is order continuous. A positive element in E is discrete if its linear span is an order ideal in E. The space E is termed discrete if the band generated by the discrete elements is the whole space. For instance, c, c_0, $l_p(1 \leqslant p \leqslant \infty)$ are discrete Banach lattices but the spaces $L^1[0,1]$ and $C[0,1]$ are not discrete. An operator T from a Banach space X into a Banach lattice E is called L-weakly compact if $\lim_n \|y_n\| = 0$ hold for every disjoint sequence $(y_n)_n$ in the solid hull of $T(B_X)$. Using Theorem 5.61 in [1], we see that every L-weakly compact operator is weakly compact, so we can say that if $T : X \to E$ is a surjective L-weakly compact operator then E is reflexive.

Theorem 2.9. *Let E be a Banach lattice such that E^a is discrete and X is a Banach space. If $T : X \to E$ is a surjective L-weakly compact operator, then E is reflexive.*

Theorem 2.10. *Let E and F be Banach lattices such that the norm of E' is order continuous and F be infinite-dimensional. If $T : E \to F$ is a regular surjective L-weakly compact operator then E' is not discrete.*

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ON THE FACTORIZATION PROPERTY AND ARENS REGULARITY OF BANACH ALGEBRAS

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Abstract. In this paper, we establish some relationships between topological centers of module actions and factorization properties and give some results in group algebras case. We also consider sufficient and necessary conditions under which the Banach algebra $A\widehat{\otimes}B$ is Arens regular. Let A and B be Banach algebras and B is unital and suppose that B is a Banach A-bimodule. In this case, if $A\widehat{\otimes}B$ is Arens regular, then A is Arens regular. In another case, assume that B is non-trivial on A and B is a unitary Banach A-bimodule. Then A and B are Arens regular if and only if $A\widehat{\otimes}B$ is Arens regular.

1. Introduction

Let B be a Banach A-bimodule, and let

$$\pi_\ell : \; A \times B \longrightarrow B \quad \text{and} \quad \pi_r : \; B \times A \longrightarrow B,$$

be the right and left module actions of A on B, respectively. Then the second dual B^{**} is a Banach A^{**}-bimodule with the following module

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actions

$$\pi_\ell^{***}:\ A^{**}\times B^{**}\longrightarrow B^{**}\quad\text{and}\quad \pi_r^{***}:\ B^{**}\times A^{**}\longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the first Arens product. Similarly, B^{**} is a Banach A^{**}-bimodule with the module actions

$$\pi_\ell^{t***t}:\ A^{**}\times B^{**}\longrightarrow B^{**}\quad\text{and}\quad \pi_r^{t***t}:\ B^{**}\times A^{**}\longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the second Arens product. Thus, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$\begin{aligned} Z^\ell_{A^{**}}(B^{**}) = Z(\pi_r) = \{b''\in B^{**}:\ &\text{the map } a''\to\pi_r^{***}(b'',a'')\\ &:\ A^{**}\to B^{**} \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.\end{aligned}$$

Suppose that A and B are Banach algebras. Since 1988 the Arens regularity of $A\widehat{\otimes}B$ has received a great deal of attention by many researchers. Among them, Ülger in [5] showed that $A\widehat{\otimes}B$ is not Arens regular, in general, even when A and B are Arens regular. He introduced a new concept of biregular mapping and showed that a bounded bilinear mapping $m: A\times B\to\mathbb{C}$ is biregular if and only if $A\widehat{\otimes}B$ is Arens regular, where $\mathbb{C}$ is the space of complex numbers. Consider the tensor product, $X\otimes Y$, of the vector space X and Y which can be constructed as a space of linear functional on $B(X\times Y)$. By $X\widehat{\otimes}Y$ we shall denote the projective tensor products of X and Y, where $X\widehat{\otimes}Y$ is the completion of $X\otimes Y$ for the norm

$$\|u\| = \inf\{\sum_{i=1}^n \|x_i\|\|y_i\|\},$$

where the infimum is taken over all the representations of u as a finite sum of the form $u=\sum_{i=1}^n x_i\otimes y_i$ [3]. The natural multiplication of $A\widehat{\otimes}B$ is the linear extension of the following multiplication on decomposable tensors $(a\otimes b)(\tilde a\otimes\tilde b) = a\tilde a\otimes b\tilde b$.

2. Main results

Let B be a Banach A-bimodule. Then B is called factors on the left (right) with respect to A, if $B=BA$ $(B=AB)$. Thus B factors on both sides, if $B=BA=AB$.

Theorem 2.1. *Suppose that B is a right Banach A-module and it has a RBAI $(e_\alpha)_\alpha\subseteq A$. Then we have the following assertions:*

(1) *B factors on the right.*

(2) *If A^* factors on the right and $Z^{\ell}_{A^{**}}(B^{**}) = B^{**}$, then B^* factors on the right.*

Theorem 2.2. *Let B be a Banach A-bimodule and A has a bounded right approximate identity. Then the following assertions hold:*

(1) $(B^*A)^{\perp} = \{b'' \in B^{**} : \ \pi_{\ell}^{***}(a'', b'') = 0 \ for \ all \ a'' \in A^{**}\}$.

(2) $(B^*A)^*$ *is isomorphism with* $\mathrm{Hom}_A(B^*, A^*)$.

Example 2.3. (1) Let G be a locally compact group. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$(L^p(G) * L^1(G))^{\perp} = \{b \in L^q(G) : \ a''b = 0 \text{ for every } a'' \in L^{\infty}(G)\},$$

and

$$(L^p(G) * L^1(G))^* \cong \mathrm{Hom}_{L^1(G)}(L^p(G), L^{\infty}(G)).$$

(2) Let G be a locally compact group. The group algebra $L^1(G)$ is a two sided ideal in $M(G)$. We know that $L^1(G)^{**} = L^1(G) \oplus C_0(G)^{\perp}$. On the other hand, $M(G)$ is a unital Banach algebra and $M(G)^{**}$ has a right identity [4, Proposition 2.9.16 (ii)] respect to the first Arens product. Then we have

$$\begin{aligned} L^1(G)^{**} &= (L^{\infty}(G)M(G))^* \oplus (L^{\infty}(G)M(G))^{\perp} \\ &\cong \mathrm{Hom}_{M(G)}(L^{\infty}(G), M(G)^*) \oplus (L^{\infty}(G)M(G))^{\perp}. \end{aligned}$$

Theorem 2.4. *Assume that B is a left Banach A-module and A has a BAI. If B^* factors on the left, then $B^{*\perp} = 0$.*

Definition 2.5. Let A be a Banach algebra and let B be a Banach A-bimodule and let $\pi : A\widehat{\otimes}B \longrightarrow B$ such that $\pi(a \otimes b) = ab$ for every $a \in A$, $b \in B$. We say that B is non-trivial on A, if π is surjective and has a bounded right inverse.

Theorem 2.6. *Let A and B be Banach algebras and B is unital. Suppose B is a Banach A-bimodule. Then*

(1) *If $A\widehat{\otimes}B$ is Arens regular, then A is Arens regular.*

(2) *If B is non-trivial on A and B be a unitary Banach A-bimodule. Then A and B are Arens regular if and only if $A\widehat{\otimes}B$ is Arens regular.*

Theorem 2.7. *Let A and B be Banach algebras and B is unital. Suppose B is a Banach A-bimodule. Then*

(1) *If $A\widehat{\widehat{\otimes}}B$ is Arens regular, then A is Arens regular.*

(2) *If B is non-trivial on A and B be a unitary Banach A-bimodule. Then A and B are Arens regular if and only if $A\widehat{\widehat{\otimes}}B$ is Arens regular.*

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2-INNER PRODUCT C^* -HILBERT MODULES AND REPRODUCING PROPERTY

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Abstract. In this paper we discuss reproducing kernels whose ranges are contained in a C^*-algebra or a Hilbert C^*-module. Using the construction of a reproducing Hilbert C^*-module associated with a reproducing kernel, we show how such a reproducing kernel can naturally be expressed in terms of operators on a Hilbert C^*-module using representations on Hilbert C^*-modules. We focus on 2-inner product kernels and prove related theorems.

1. Introduction

The theory of reproducing kernels is fundamental and is applicable widely in mathematics. The abstract theory of reproducing kernel Hilbert spaces has been developed over a number of years inside the domain of physics, and the reproducing kernel Hilbert space theory provides a unified framework for stochastic processes and signal processing. Reproducing kernel Hilbert space method provides a rigorous and effective framework for smooth multivariate interpolation.

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2. Preliminaries

When one considers Hilbert C^*-modules, one should be rather careful at certain points concerning the existence of adjoints for operators on a Hilbert C^*-module and the self-duality of Hilbert C^*-modules. In following, we define Hilbert $\mathcal{A}$-module.

Definition 2.1. A Hilbert C^*-module over a C^* -algebra $\mathcal{A}$ is a right $\mathcal{A}$-module E equipped with $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ which is conjugate $\mathcal{A}$-linear in the first variable and $\mathcal{A}$-linear in the second variable such that E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

Example 2.2. Let X be a locally compact Hausdorff space and H a Hilbert space, the Banach space $C_0(X, H)$ of all continuous H-valued functions vanishing at infinity is a Hilbert C^*-module over the C^*-algebra $C_0(X)$ with inner product $\langle f, g\rangle(x) := \langle f(x), g(x)\rangle$ and module operation $(f\varphi)(x) = f(x)\varphi(x)$, for all $f \in C_0(X, H)$ and $\varphi \in C_0(X)$. Every C^*-algebra $\mathcal{A}$ is a Hilbert C^*-module over itself with inner product $\langle a, b\rangle := a^*b$.

Definition 2.3. Let $\mathcal{B}$ be a C^*-algebra. A kernel $k : S \times S \to \mathcal{B}$ is positive definite if for every $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in S$ the matrix $[k(s_i, s_j)]_{i,j=1}^n$ is positive in $M_n(\mathcal{B})$.

It follows from the definition that a kernel $k : S \times S \to \mathcal{B}$ is positive definite if and only if for all $s_1, \ldots, s_n \in S$ and $b_1, \ldots, b_n \in \mathcal{B}$, the sum $\sum_{i,j=1}^n b_i^* k(s_i, s_j) b_j$ is positive in $\mathcal{B}$. If a kernel k from $S \times S$ into $L_{\mathcal{A}}(X)$ can be written in the form

$$k(s, t) = v(s)^* v(t) \qquad (s, t \in S),$$

where v is a map from S to $L_{\mathcal{A}}(X, X_v)$ for some Hilbert $\mathcal{A}$-module X_v, then k is automatically positive definite. Such a map v is called the Kolmogorov decomposition for k. Conversely, every positive definite kernel k with values in $L_{\mathcal{A}}(X)$ has an essentially unique minimal Kolmogorov decomposition.

Consider a Hilbert $\mathcal{A}$-module of X-valued functions spanned by those of the form $s \mapsto k(s, t)x$, for some $t \in S$ and some $x \in X$. We denote by E_0 the set of all X-valued functions on S having finite support and by E the set of all X-valued functions on S. We can identify E with a subspace of the algebraic antidual E_0' of E_0 by defining the pairing of E and E_0 by

$$(g, f) = \sum_{s \in S} \langle g(s), f(s)\rangle_X \qquad (g \in E, f \in E_0).$$

Given a kernel $k : S \times S \to L_{\mathcal{A}}(X)$, we can define the associated convolution operator $\tilde{k} : E_0 \to E$ by

$$(\tilde{k}f)(s) = \sum_{s \in S} k(s,t)f(t) \qquad (s \in S).$$

Then the kernel k is positive definite if and only if its associated convolution operator $\tilde{k} : E_0 \to E$ is positive, that is, $(\tilde{k}f, f) \geq 0$ for all $f \in E_0$.

3. Main Results

In this section, we extend the concept of 2-inner product to C^*-modules and express reproducing property on these spaces.

Definition 3.1. Let E be a Hilbert C^*-module over a C^* -algebra $\mathcal{A}$ of dimension greater than 1. The function $\langle ., .; . \rangle : E \times E \times E \to \mathcal{A}$ is called a 2-inner product if the following conditions hold:

(1) $\langle x, x; z \rangle \geq 0$ and $\langle x, x; z \rangle = 0$ iff x and z are linearly dependent.
(2) $\langle x, x; z \rangle = \underline{\langle z, z; x \rangle}$.
(3) $\langle y, x; z \rangle = \overline{\langle x, y; z \rangle}$.
(4) $\langle \alpha x, y; z \rangle = \alpha \langle x, y; z \rangle$, for all scalars $\alpha \in F$.
(5) $\langle x_1 + x_2, y; z \rangle = \langle x_1, y; z \rangle + \langle x_2, y; z \rangle$.

Then the pair $(E, \langle ., .; . \rangle)$ is called a 2-inner product Hilbert C^*-module.

Example 3.2. Let $C_0(X, H)$ be the Hilbert C^*-module as explained in Example 2.2. We define a 2-inner product over $C_0(X, H)$ by

$$\langle f, g; h \rangle = \|h\|^2 \int \overline{g(x)} f(x) dx - \int \overline{h(x)} f(x) dx. \int \overline{g(x)} h(x) dx$$

Definition 3.3. Let S be a nonempty set and $\mathcal{A}$ be a C^*-algebra. Suppose E is a 2-inner product Hilbert C^*-module over $\mathcal{A}$ and $\mathcal{E}$ is a subspace of E such that every evaluation on $\mathcal{E}$ is bounded. In this case, there exists a positive kernel in terms of Definition 2.3 such that for every $x \in S$

$$\mathrm{Eval}(f)(x) = f(x) \qquad (x \in S, f \in E). \tag{3.1}$$

This $\mathcal{E}$ is called a reproducing kernel $\mathcal{A}$-module.

Suppose $\mathcal{E}^\dagger$ is a subspace of E included all two variable functions of $S \times S$ and endowed with a 2-inner product. Let every evaluation on $\mathcal{E}^\dagger$ be bounded, that is, instead of equation 3.1, we have

$$\mathrm{Eval}(x, y) = f(x, y) \qquad (x, y \in S, f \in E). \tag{3.2}$$

Then $\mathcal{E}^\dagger$ is called a 2-inner product reproducing kernel $\mathcal{A}$-module.

As an example of a 2-inner product reproducing kernel C^*-module, we refer the reader to [1, Example 3.2].

In following, we explain some properties of 2-inner product reproducing kernel C^*-modules and prove two important theorems.

Proposition 3.4. *If $\mathcal{E}^\dagger$ is a 2-inner product reproducing kernel C^*-module with reproducing kernel $k(x, y, z)$, then $k(x, y, x) = \overline{k(y, x, y)}$.*

In next theorem, we extend Kolmogorov decomposition to 2-inner product reproducing kernel C^*-modules.

Theorem 3.5. *For a positive 2-inner product kernel k from $S \times S \times S$ into $L_{\mathcal{A}}(X)$, there exists a map v from S to $L_{\mathcal{A}}(X, X_v)$ for some Hilbert $\mathcal{A}$-module X_v such that*

$$k(x, y, z) = v(x)^* v(y) \|v(z)\| \qquad (z, \|v(z)\| \neq 0). \tag{3.3}$$

Equation 3.3 defines an extension of Kolmogorov decomposition. In particular cases, we can choose v as a unitary mapping. The next theorem explains an application of this kind of decomposition.

Theorem 3.6. *Let $\mathcal{A}$ be a C^*-algebra, E be a Hilbert $\mathcal{A}$-module and $\mathcal{E}^\dagger$ be a 2-inner product $\mathcal{A}$- module. Then there exists a faithful representation π of $\mathcal{A}$ on a Hilbert space H and an isometric, linear isomorphism v from $\mathcal{E}^\dagger$ onto a concrete Hilbert $\pi(\mathcal{A})$-module F of operators from H to a Hilbert space K such that*

$$\langle v(x), v(y); v(z)\rangle = \pi(\langle x, y; z\rangle) \qquad (x, y, z \in \mathcal{E}^\dagger).$$

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COUPLED FIXED POINT THEOREMS $\alpha_* - \psi-$CONTRACTIVE ON PARTIALLY ORDERED METRIC SPACE

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Abstract. In 1987, Guo and Lakshmikantham introduced the notion of coupled fixed points. There exist a lot of works done on the coupled fixed points of some types of mappings. Recently, the notion of $\alpha - \psi-$contractive type mappings has been introduced by Samet et al. in 2012. In this paper, we introduce $\alpha_* - \psi-$contractive multifunctions on a partially ordered metric space and provide some results about coupled fixed points without upper semi-continuity condition.

1. Introduction

In this paper, we introduce generalized $\alpha_* - \psi-$contractive multifunctions and present some results about coupled fixed points of these functions. Denote by Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$ [2].

Let (X, d) be a metric space and T a self map on X. Then T is called an $\alpha - \psi-$contraction mapping whenever there exist $\psi \in \Psi$ and

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$\alpha : X \times X \to [0, +\infty)$ such that $\alpha(x, y)d(Tx, Ty) \leq (d(x, y))$ for all $x, y \in X$ [2]. Also, we say that T is $\alpha-$admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [2]. Also, we say that space X has the property B_α if for any sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to x$, we have $\alpha(x_n, x) \geq 1$ for all $n \geq 1$ [2]. Let (X, d) be a complete metric space and T an $\alpha-$admissible $\alpha - \psi-$contractive mapping on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If T is continuous or space X has the property B_α, then T has a fixed point (see Theorems 2.1 and 2.2 in [2]). Finally, we say that X has the property H_α whenever for each $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. If X has the property H_α in Theorems 2.1 and 2.2 in [2], then X has a unique fixed point ([2], Theorem 2.3).

Let (X, d) be a metric space and $T : X \twoheadrightarrow X$ a multifunction on X. In this case, we say that T is a $\alpha_* - \psi-$contractive multifunction if there exist $\psi \in \Psi$ and $\alpha : X \times X \to [0, +\infty)$ such that $\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where H is the Hausdorff metric and $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ for all $A, B \subseteq X$. Also, we say that T is α_*-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, see [1].

2. Preliminaries

We collect some basic definitions, lemmas and notations, which will be used throughout the paper. We denote by $CB(X)$ the family of nonempty, closed and bounded subsets of X and by $CL(X)$ the class of nonempty closed subsets of X. For $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \to \mathbb{R}^+$ by $H(A, B) = \max\{\delta(A, B), \delta(B, A)\}$, where $\delta(A, B) = \sup\{D(a, B) : a \in A\}, \delta(B, A) = \sup\{D(b, A) : b \in B\}$ with $D(a, C) = \inf\{d(a, x) : x \in C\}$. Note that H is called the Pompeiu-Hausdroff metric induced by the metric d.

Definition 2.1. Let (X, d) be a metric space and $T : X \to CB(X)$ be a given multifunction. Let $\alpha : X \times X \to [0, +\infty)$, $\psi \in \Psi$ and $m(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\}$ for all $x, y \in X$. We say that T is a generalized $\alpha_* - \psi-$contractive multifunction whenever $\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(m(x, y))$ for all $x, y \in X$.

Definition 2.2. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that $D(B, y) = d(z, y)$.

Definition 2.3. A multi-valued mapping $T : X \longrightarrow CB(X)$ is said to have approximate values in X if Tx is an approximation for each $x \in X$.

Definition 2.4. Let $(X, \preceq, d)$ be a partially ordered complete metric space, a multi-valued mapping $T : X \to CB(X)$ is called order closed if for monotone sequences $x_n \in X$ and $y_n \in Tx_n$, with $x_n \to x$ and $y_n \to y$, $y \in Tx$.

Definition 2.5. Let $F : X \times X \to CB(X)$ be a multifunction, where (X, d) is a metric space. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if $x \in F(x, y)$, $y \in F(y, x)$.

3. Main results

Let (X, d) be a metric space and $F : X \times X \to X$ a mapping. We define $\delta((x, y), (u, v)) = d(x, u) + d(y, v)$ and

$$\begin{aligned} m((x,y),(u,v)) = \max\{&\delta((x,y),(u,v)), \delta((x,y),(F(x,y),F(y,x))), \\ &\delta((u,v),(F(u,v),F(v,u))), \frac{1}{2}[\delta((x,y),(F(u,v),F(v,u))) \\ &+ \delta((u,v),(F(x,y),F(y,x)))\} \end{aligned}$$

for all (x, y), $(u, v) \in X \times X$. Also, if $F : X \times X \to CB(X)$ is a multifunction mapping we define $\Delta(T_F(x, y), T_F(u, v))$ $= H(F(x, y), F(u, v)) + H(F(y, x), F(v, u))$ and

$$\begin{aligned} M((x,y),(u,v)) = \max\{&\delta((x,y),(u,v)), \Delta((\{x\},\{y\}),T_F(x,y)), \\ &\Delta((\{u\},\{v\}),T_F(u,v)), \frac{1}{2}[\Delta((\{x\},\{y\}),T_F(u,v)) \\ &+ \Delta((\{u\},\{v\}),T_F(x,y))\} \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$.

Now, we are ready to state and prove our main results.

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. For two subsets A, B of X one marks $A \preceq_r B$ if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 3.2. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \twoheadrightarrow X$ be a given multifunction mapping. We say that F has mixed monotone property if for all $x, y \in X$ we have: (1) for all $x_1, x_2 \in X$, $x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq_r F(x_2, y)$ and (2) for all $y_1, y_2 \in X$, $y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq_r F(x, y_1)$.

Theorem 3.3. *Let (X, d) be a complete metric space and $F : X \times X \to CB(X)$ be a given multifunction mapping and an approximation. Suppose that there exist $\psi \in \Psi$, a function $\alpha : CB(X) \times CB(X) \to [0, +\infty)$, and $\alpha_* : CB(X^2) \times CB(X^2) \to [0, +\infty)$ which satisfy the following conditions:*

(i) *For all* $(x, y), (u, v) \in X \times X$,

$$\alpha_*(T_F(x, y), T_F(u, v))H(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(M((x, y), (u, v))),$$

(ii) *For all* $(x, y), (u, v) \in X \times X$, *we have*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha_*(T_F(x, y), T_F(u, v)) \geq 1.$$

(iii) *There exists* $(x_0, y_0) \in X \times X$ *with* $\alpha_*(\{(x_0, y_0)\}, T_F(x_0, y_0)) \geq 1$ *and* $\alpha_*(T_F(y_0, x_0), \{(y_0, x_0)\}) \geq 1$.

(iv) *If* $\{x_n\}$, $\{y_n\}$ *are sequences in* X *with* $\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$, $\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1$, $x_n \to x \in X$ *and* $y_n \to y \in X$ *as* $n \to +\infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *and* $\{y_{n(k)}\}$ *of* $\{y_n\}$ *such that* $\alpha((x_{n(k)}, y_{n(k)}), (x, y)) \geq 1$ *and* $\alpha((y, x), (y_{n(k)}, x_{n(k)})) \geq 1$ *for all* $k \geq 1$.

Then, F *has a coupled fixed point, that is, there exists* $(x^*, y^*) \in X \times X$ *such that* $x^* \in F(x^*, y^*)$ *and* $y^* \in F(y^*, x^*)$.

Theorem 3.4. *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \twoheadrightarrow X$ *be a multifunction mapping having the mixed monotone property on* X. *Assume that there exists* $k \in [0, 1)$ *such that* $H(F(x, y), F(u, v)) \leq (k/2)M((x, y), (u, v))$ *for all* $x, y, u, v \in X$ *such that* $x \succeq u$ *and* $y \preceq v$. *Also, assume that* X *has the following properties:*

(i) *If for a nondecreasing sequence* $\{x_n\}$, $x_n \to x \in X$, *then* $x_n \preceq x$ *for all* n.

(ii) *If for a nonincreasing sequence* $\{y_n\}$, $y_n \to y \in X$, *then* $y_n \succeq y$ *for all* n.

(iii) *There exists* $(x_0, y_0) \in X \times X$ *such that* $\{x_0\} \preceq_r F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq_r \{y_0\}$.

Then F *has a coupled fixed point.*

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COMMON FIXED POINTS OF MULTI-VALUED MAPPINGS WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. Recently, B. Samet, C. Vetro and P. Vetro introduced the notion of $\alpha - \psi-$contractive type mappings and established some fixed point theorems for the mappings in complete metric spaces. In this paper, with the motivation of $\alpha-\psi-$contractiveness of a mapping T on a metric space, we introduce a new notion of coupled α_*-g-contractiveness of two mappings T and S on a metric space. Then, considering this notion, some common fixed point results have been obtained. Moreover, some examples and applications to integral equations are given to illustrate the usability of the obtained results.

1. Introduction

It is well know that fixed point theory has many applications and was extended by several authors from different views (see, for example, [3]). Recently, B. Samet et al introduced the notion of $\alpha - \psi-$contractive type mappings [2]. We denote by Ψ the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the $n-$th iterate of ψ. It is known that, if $\psi \in \Psi$ then

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$\psi(t) < t$ for all $t > 0$; [2]. Let (X, d) be a metric space, T a self-map on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ a function. Then T is called a $\alpha - \psi-$contraction mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Also, we say that T is $\alpha-$admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$; [2]. The aim of this paper is to introduce the notion of coupled $\alpha_* - g-$ contractive multi-valued mappings, weakly increasing multi-valued mapping and gives a common fixed point result about the multi-valued mappings.

2. Preliminaries

We collect some basic definitions, lemmas and notations which will be used throughout the paper. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and $\mathbb{N}$ denote the set of positive integers. Also, we denotes $CB(X)$ the family of nonempty, closed and bounded subsets of X and by $CL(X)$ the class of nonempty closed subsets of X. Also, we denote by H the Pompeiu-Hausdroff metric on $CB(X)$ induced by the metric d;

$$H(A, B) = \max\{\sup_{y \in B} D(A, y), \sup_{x \in A} D(x, B)\}$$

and $D(x, B) = \inf\{d(x, y) : y \in B\}$.

Definition 2.1. Let $g : [0, +\infty)^5 \to [0, +\infty)$ be a continuous function and satisfy the following conditions:

(a) $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) = h \in (0, 1)$.

(b) g is sub-homogenous, that is,

$$g(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \leq \lambda g(x_1, x_2, x_3, x_4, x_5)$$

for all $\lambda \geq 0$ and all $(x_1, x_2, x_3, x_4, x_5) \in [0, +\infty)^5$.

(c) If $x_i, y_i \in [0, +\infty)$ and $x_i < y_i$, for $i = 1, 2, 3, 4$ then

$$g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0)$$

and

$$g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4).$$

In this case, we write $g \in \mathcal{R}$.

Definition 2.2. Let (X, d) be a metric space and $T, S : X \to CB(X)$ be given coupled multi-valued mappings. Let $\alpha : X \times X \to [0, +\infty)$, $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ for all $A, B \subset X$. One says that

T, S are coupled $\alpha_* - g-$contractive multi-valued mappings if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $g \in \mathcal{R}$ such that

$$\alpha_*(Ax, By)H(Ax,By) \leq \qquad (2.1)$$
$$g(d(x, y), D(x, Ax), D(y, By), D(x, By), D(y, Ax))$$

for all $x, y \in X$, where A and B can be S or T.

Definition 2.3. Let $T, S : X \to CB(X)$ and $\alpha : X \times X \to [0, +\infty)$. One says that T, S are α_*-common admissible if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha_*(Ax, By) \geq 1 \qquad (2.2)$$

$A, B = T$ or S for all $x, y \in X$.

Definition 2.4. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that

$$D(B, y) = d(z, y).$$

Definition 2.5. A multi-valued mapping $T : X \longrightarrow CB(X)$ is said to have approximate values in X if Tx is an approximation for each $x \in X$.

Definition 2.6 ([1])**.** Let $(X, \preceq, d)$ be a partially ordered complete matric space, a multi-valued mapping $T : X \to CB(X)$ is called order closed if for monotone sequences $x_n \in X$ and $y_n \in Tx_n$, with $x_n \to x$ and $y_n \to y$, we have $y \in Tx$.

3. Main results

Now, we are ready to state and prove our main results.

Lemma 3.1. *If $g \in \mathcal{R}$ and $u, v \in [0, +\infty)$ are such that*

$$u \leq \max\{g(v, v, u, v + u, 0), g(v, v, u, 0, v + u),$$
$$g(v, u, v, v + u, 0), g(v, u, v, 0, v + u)\}.$$

Then there is $h \in (0, 1)$ with $u \leq hv$.

Lemma 3.2. *Let (X, d) be a metric space. Suppose that $T, S : X \to CB(X)$ are coupled $\alpha_* - g-$contractive multi-valued mappings. Then* $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

Theorem 3.3. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to CB(X)$ are coupled $\alpha_* - g-$contractive multi-valued mappings and satisfies the following conditions:*

(i) *T, S are α_*-common admissible.*
(ii) *There exists $x_0 \in X$ such that $\alpha_*(\{x_0\}, Tx_0) \geq 1$.*
(iii) *T or S is order closed.*

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of T and S.

Fixed point theorems for operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations.

Theorem 3.4. *Let $X = C([a,b], \mathbb{R}^n), a > 1$ and $d : X \times X \to \mathbb{R}$ be defined as follows:*

$$d(x,y) = \max_{t\in[a,b]} ||x(t) - y(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + s(t),\ y(t) = \int_a^b K_2(t,s,y(s))ds + h(t),$$

where $t \in [a,b] \subset \mathbb{R}, x, y, s, h \in X$.

Suppose that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_y \in X$, for each $x, y \in X$, where,

$$F_x(t) = \int_a^b K_1(t,s,x(s))ds, \quad G_y(t) = \int_a^b K_2(t,s,y(s))ds,$$

for all $t \in [a,b]$. If there exists $0 < h < 1$ such that for every $x, y \in X$

$$||F_x(t) - G_y(t) + s(t) - h(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a \le hR(x,y)(t),$$

where

$$R(x,y)(t) \in \{A(x,y)(t), B(x,y)(t), C(x,y)(t), D(x,y)(t), E(x,y)(t)\},$$
$$A(x,y)(t) = ||x(t) - y(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a,$$
$$B(x,y)(t) = ||F_x(t) + s(t) - x(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a,$$
$$C(x,y)(t) = ||G_y(t) + h(t) - y(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a,$$

$$D(x,y)(t) = \frac{(||G_y(t) + h(t) - x(t)||_\infty + ||F_x(t) + s(t) - y(t)||_\infty)\sqrt{1+a^2} \cos \tan^{-1} a}{2},$$

$$E(x,y)(t) = \frac{||G_y(t) + h(t) - x(t)||_\infty ||F_x(t) + s(t) - y(t)||_\infty \sqrt{1+a^2} \cos \tan^{-1} a}{1 + \max_{t\in[a,b]} A(x,y)(t)}.$$

Then the system of Urysohn integral equations have a unique common solution.

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MULTIPLICATION BY A BLASCHKE PRODUCT WITH TWO ZEROS

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Abstract. In this paper we generalize the results obtained in case of multiplication operators on $L^2_a(\mathbb{D})$ induced by Blaschke products with two zeros in $\mathbb{D}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Bergman space $L^2_a(\mathbb{D})$ is the Hilbert space of analytic functions f in $\mathbb{D}$ such that $\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty$, where dA is the normalized area measure of $\mathbb{D}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two functions in $L^2_a(\mathbb{D})$, then the inner product of f and g is given by

$$< f, g >= \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

We study multiplication operators on $L^2_a(\mathbb{D})$ induced by analytic functions. Thus for $\varphi \in H^\infty(\mathbb{D})$, the space of bounded analytic functions in $\mathbb{D}$, we define $M_\varphi : L^2_a(\mathbb{D}) \longrightarrow L^2_a(\mathbb{D})$ by $M_\varphi f = \varphi f,\ f \in L^2_a(\mathbb{D})$. It is easy to check that M_φ is a bounded linear operator on $L^2_a(\mathbb{D})$ with $\|M_\varphi\| = \|\varphi\|_\infty = \sup\{|\varphi(z)| : \ z \in \mathbb{D}\}$.

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2. Main results

We begin with a simple method for constructing reducing subspaces of certain analytic multiplication operators. The main reference for this line of ideas is [4]. For any $\lambda \in \mathbb{D}$ recall that φ_λ is the Mobius map on $\mathbb{D}$ defined by $\varphi_\lambda(z) = \dfrac{\lambda - z}{1-\overline{\lambda}z}$, $z \in \mathbb{D}$. We define a composition operator C_λ and a unitary U_λ on $L_a^2(\mathbb{D})$ as follows:

$$C_\lambda f = f \circ \varphi_\lambda,\ U_\lambda f = f \circ \varphi_\lambda k_\lambda,\ f \in L_a^2(\mathbb{D}).$$

Here k_λ is the normalized reproducing kernel of $L_a^2(\mathbb{D})$ at λ. It is easy to check that U_λ^2 is the identity operator. Thus U_λ is a self-adjoint unitary operator whose spectrum consists of 1 and - 1. By the spectral theorem there exists an orthogonal projection p_λ with range space M_λ such that $U_\lambda = P_\lambda - P_\lambda^\perp$, where $\lambda^\perp$ is the orthogonal projection onto $M_\lambda^\perp$. In fact $P_\lambda = \dfrac{1}{2}(I + U_\lambda)$, $P_\lambda^\perp = \dfrac{1}{2}(I - U_\lambda)$. Furthermore, M_λ is simply the set of fixed points of U_λ and $M_\lambda^\perp$ is the set of fixed points of $-U_\lambda$; see [4].

Lemma 2.1. *Let $\varphi \in H^\infty(\mathbb{D})$ and $\lambda \in \mathbb{D}$. If φ is a fixed point of C_λ, then M_λ (and hence $M_\lambda^\perp$) is reducing subspace of $M_\varphi : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$.*

It was shown in [4] that

$$M_\lambda = U_\sigma(L_{ae}^2(\mathbb{D})) = \{f \circ \varphi_\sigma k_\sigma : f \in L_{ae}^2(\mathbb{D})\}$$

and

$$M_\lambda^\perp = U_\sigma(L_{a0}^2(\mathbb{D})) = \{f \circ \varphi_\sigma k_\sigma : f \in L_{a0}^2(\mathbb{D})\}$$

where σ is the geodesic midpoint between 0 and λ:

$$\sigma = \frac{1-\sqrt{1-|\lambda|^2}}{|\lambda|^2}\lambda;$$

if $\lambda = 0$, then σ is 0, which is the limit of the above expression as $\lambda \to 0$. The fixed points of C_λ and $-C_\lambda$ were also described in [4]in terms of even and odd functions and Mobius maps. We will not need this description of fixed points, but the following corollary serves as an interesting example.

Lemma 2.2. *If $\lambda \in \mathbb{D}$ and $\varphi(z) = z\varphi_\lambda(z)$, then M_λ (and hence $M_\lambda^\perp$) is a reducing subspace of $M_\varphi : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$.*

Proof. It is clear that φ is fixed point of C_λ. ☐

We now consider multiplication operators on $L^2_a(\mathbb{D})$ induced by Blaschke products with two zero in $\mathbb{D}$ show how the above ideas can be used to produce reducing subspaces for such operators; the corollary above is a critical special case.

Lemma 2.3. *For any two points a and b in $\mathbb{D}$ there exists a unimodulus constant $c \in \mathbb{C}$ such that $\varphi_b \circ \varphi_a = c\varphi_\lambda$, where $\lambda = \varphi_a(b)$.*

Proof. This is well known; see [4]. ☐

Fix two points a and b (not necessarily distinct) in $\mathbb{D}$ and let $\varphi = \varphi_a\varphi_b$ be the Blaschke product with zero a and b. We can write $\varphi = \psi \circ \varphi_a$, where

$$\psi(z) = z\varphi_b \circ \varphi_a(z), \ z \in \mathbb{D}.$$

By Lemma 2.2 there exists a unimodulus constant c such that

$$\varphi_b \circ \varphi_a(z) = c\varphi_\lambda(z), \ z \in \mathbb{D}$$

wrhere $\lambda = \varphi_a(b)$. Thus $\psi(z) = c\varphi_\lambda(z)$, and by Lemma 2.3, the spaces M_λ and $M_\lambda^\perp$ are reducing subspaces of M_ψ.

Since the operator U_a establishes a unitary equvalence between M_φ and M_ψ, we conclude that

$$U_a(M_\lambda) = \{f \circ \varphi_a k_a : f \in M_\lambda\}$$

and

$$U_a(M_\lambda^\perp) = \{f \circ \varphi_a k_a : f \in M_\lambda^\perp\}$$

are reducing subspaces of M_φ .

Let m be the geodisc midpoint between a and b. Then $\sigma = \varphi_a(m)$ is the geodesic midpoint between 0 and $\lambda = \varphi_a(b)$. Recall that

$$M_\lambda = \{f \circ \varphi_\sigma k_\sigma : f \in L^2_{ae}(\mathbb{D})\}$$

and

$$M_\lambda^\perp = \{f \circ \varphi_\sigma k_\sigma : f \in L^2_{a0}(\mathbb{D})\}.$$

Combining these formulae with the conclusions reached in the previous paragraph, we obtain the following reducing subspaces for M_φ:

$$X_e = \{f \circ \varphi_a \circ \varphi_a k_\sigma \circ \varphi_a k_a : f \in L^2_{ae}(\mathbb{D})\}$$

and

$$X_0 = \{f \circ \varphi_a \circ \varphi_a k_\sigma \circ \varphi_a k_a : f \in L^2_{a0}(\mathbb{D})\}.$$

We proceed to show that these are the only two proper reducing subspaces of M_φ on $L^2_a(\mathbb{D})$.To this end we will need the following well known fact.

Lemma 2.4. *Suppose that φ is any function in $H^\infty(\mathbb{D})$ with $\|\varphi\|_\infty \leq 1$. Then any invariant (or reducing) subspace of M_φ is also an invariant (or reducing) subspace of $M_{h\circ\varphi}$, where h is any function in $H^\infty(\mathbb{D})$.*

Theorem 2.5. *Let a and b be two points in $\mathbb{D}$ and let m the geodesic midpoint between a and b. Then*

$$X_e = \{f \circ \varphi_m k_m : f \in L^2_{ae}(\mathbb{D})\}$$

and

$$X_0 = \{f \circ \varphi_m k_m : f \in L^2_{a0}(\mathbb{D})\}$$

are the only proper reducing subspaces of M_φ, where $\varphi = \varphi_a\varphi_b$.

Finally in this section we present a description of the commutant of M_φ on $L^2_a(\mathbb{D})$ when φ is a Blaschke product with two zeros.

Theorem 2.6. *Let a and b be two points in $\mathbb{D}$ with geodesic midpoint m. Let $\varphi = \varphi_a\varphi_b$. Then the commutant of M_φ consists of operators T which admit the following representation: $Tf = Ff_1 + Gf_2/(z-m)$, $f \in L^2_a(\mathbb{D})$, where F and G are functions in $H^\infty(\mathbb{D})$ and $f = f_1 + f_2$ with $f_1 \in X_e$ and $f_2 \in X_0$.*

A similar description of the commutant of M_φ on Bergman space for $\varphi = \varphi_a\varphi_b$ is obtained in [2]. More specifically, the commutant of M_φ is described in [2] and [4] as the set of operators of the form $T = M_F + M_G C_r + \lambda M_{(z-c)^{-1}}(I - C_r)$, where F and G are function in $H^\infty(\mathbb{D})$, λ is any complex constant,

$$r = \frac{a + b - |a|^2 b - |b|^2 a}{1 - |ab^2}, \qquad c = \frac{1 - \sqrt{1 - |r|^2}}{\overline{r}}$$

and C_r is the composition operator induced by the Mobius map φ_r. Although this description seemingly suggests that $(M_\varphi)'$ can be identified with $H^\infty(\mathbb{D}) \oplus H^\infty(\mathbb{D}) \oplus \mathbb{C}$, while our Theorem 2.5 suggests that $(M_\varphi)'$ is simply $H^\infty(\mathbb{D}) \oplus H^\infty(\mathbb{D})$, it can be checked that the two are equivalent. In fact, the equivalence becomes apparent when $a = b = 0$.

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THE COMMUTANT OF MULTIPLICATION OPERATORS

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Abstract. In this paper we consider the Hilbert space of analytic functions on a plane domain Ω and multiplication operators on such spaces induced by functions in $H^\infty(\Omega)$. Recently, K. Zhu has given condition under which the adjoints of multiplication operators on Hilbert spaces of analytic functions belong to the Cowen-Douglas classes. In this paper, we provide some sufficient conditions which give the converse of the main result obtained by K. Zhu. We also characterize the commutant of certain multiplication operators.

1. Introduction

Let H be a Hilbert space of functions that are analytic on a plane domain Ω such that:

(1) Every point $\omega \in \Omega$ is a nonzero bounded functional on H, so that H has a reproducing kernel k_ω such that $f(\omega) =< f, k_\omega >$ for all $f \in H$.
(2) If f is function in H, then so is zf.
(3) If $f \in H$ and $f(\lambda) = 0$, then there is a function $g \in H$ such that $(z-\lambda)g = f$.

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A space H satisfying the above conditions is called a Hilbert space of analytic functions on Ω. The Hardy and Bergman spaces are examples of Hilbert spaces of analytic functions on the open unit disk.

A complex-valued functions ϕ on Ω for which $\phi f \in H$ for every $f \in H$ is called a multiplier of H, and every multiplier ϕ on H determines a multiplication operator M_ϕ on H by $M_\phi f = \phi f,\ f \in H$. The set of all multipliers of H is denoted by $M(H)$. Clearly $M(H) \subset H^\infty(\Omega)$, where $H^\infty(\Omega)$ is the space of all bounded analytic functions on Ω. In fact $\|\phi\|_\infty \leq \|M_\phi\|$. A good source on this topic is [3].

Recall that for a positive integer n and a domain $U \subset \mathbb{C}$, the Cowen-Douglas class $B_n(U)$ consists of bounded linear operators T on any fixed separable infinite-dimensional Hilbert space X with the following properties:

(a) $\operatorname{ran}(\lambda - T) = X$ for every $\lambda \in U$.
(b) $\dim(\ker(\lambda - T)) = n$ for every $\lambda \in U$.
(c) $\operatorname{span}\{\ker(\lambda - T) : \lambda \in U\} = X$.

Here span denote the closed linear span of a collection of sets in X. For a study of the Cowen-Douglas classes B_n , we mention [1].

Also recall that a bounded linear operator A on a Hilbert space is a Fredholm operator if and only if ranA is closed and both kerA and $\ker A^*$ are finite-dimensional. We use $\sigma(A)$ and $\sigma_e(A)$ to denote, respectively, the specturm and the essential spectrum of A.

We show that a lot of multiplication operators on Hilbert spaces of analytic functions belong to the Cowen-Douglas B_1 and hence are irreducible. Our analysis also reveals an interesting fact about the Cowen-Douglas classes: an operator may simultaneously belong to B_n for several different n.

Proposition 1.1. *Suppose $\varphi \in H^\infty(\Omega)$ and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \bigcap \varphi^{-1}(z)$ is a singleton for every $z \in V$. Then the adjoint of the operator $M_\varphi : H \to H$ belongs to the Cowen-Douglas class $B_1(U)$, where $U = \{\overline{z} : z \in V\}$.*

Proposition 1.2. *Suppose $\varphi \in H^\infty(\Omega)$ and V is a (nonempty) domain contained in $\varphi(\Omega)$. If there exists a positive integer n such that $\Omega \bigcap \varphi^{-1}(z)$ consists of n points (counting multiplicity) for every $z \in V$, then the adjoint of the operator $M_\varphi : H \to H$ belongs to the Cowen-Douglas class $B_n(U)$, where $U = \{\overline{z} : z \in V\}$.*

2. Main results

In Propositions 1.1 and 1.2, sufficient conditions for the adjoint of a multiplication operator on Hilbert spaces of analytic functions has been given to belong to the Cowen-Duglas class B_n for a positive integer n. We investigate the converse of Zhu's results. Also we consider the commutant of special multiplication operators.

In the rest, we assume that H is a Hilbert spaces of analytic functions on a bounded plane domain Ω. We further assume that

$$M(H) = H^\infty(\Omega).$$

In the following by k_λ we mean the unit vector $k_\lambda/\|k_\lambda\|$.

Theorem 2.1. *Suppose $\phi \in H^\infty(\Omega)$ and V is a domain in $\mathbb{C}$ such that, for a positive integer n, the adjoint of the operator $M_\phi : H \to H$ belongs to the Cowen-Douglas class $B_n(U)$, where $U = \{\overline{z} : z \in V\}$. Also suppose that the convergence of any sequence $\{z_n\}_n \subset \Omega$ to a boundary point of Ω implies the weak convergence of $\{K_{z_n}\}_n$. Then $V \subset \phi(\Omega)$ and $\Omega \bigcap \phi^{-1}(\lambda)$ consists of n points (counting multiplicity) for every $\lambda \in V$.*

In the following, let Ω be such that if $\lambda \in \Omega$ then $-\lambda \in \Omega$. Also we assume that the composition operator $C_{-z} : H \to H$ defined by $C_{-z}f = f(-z)$ is bounded.

Proposition 2.2. *Suppose $\phi \in H^\infty(\Omega)$ and there exists a domain and there exists a domain $V \subset \phi(\Omega)$ such that $\Omega \bigcap \phi^{-1}(\omega)$ is a singleton for every $\omega \in V$. If ϕ is odd and $SM_\phi = -M_\phi S$, then $S = M_h C_{-z}$ for some $h \in H^\infty(\Omega)$.*

Theorem 2.3. *Let $\phi \in H^\infty(\Omega)$ be an odd map and suppose that for a domain $V \subset \phi(\Omega)$ the set $\Omega \bigcap \phi^{-1}(\omega)$ is a singleton for every $\omega \in V$. If $SM_{\phi^2} = M_{\phi^2} S$ and $SM_\phi - M_\phi S$ is compact, then $S = M_h$ for some $h \in H^\infty(\Omega)$.*

Proof. Clearly we can see that $TM_\phi = -M_\phi T$, where $T = SM_\phi - M_\phi S$. So by Proposition 2.2, there exists $\psi \in H^\infty(\Omega)$ such that $T = M_\psi C_{-z}$. Since T is compact, M_ψ is also compact, and by the Fredholm alternative, $\psi = 0$. Thus $SM_\phi = M_\phi S$ and by Proposition 8 in [4], $S = M_h$ for some $h \in H^\infty(\Omega)$ and so the proof is complete. $\square$

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HAUSDORFF–YOUNG INEQUALITY FOR THE SHORT-TIME FOURIER TRANSFORM

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Abstract. The aim of this note is to prove a variant of the celebrated Hausdorff–Young inequality for the short-time Fourier transform.

1. Introduction and preliminaries

The *short-time Fourier transform* (STFT) was first introduced by D. Gabor in [2] to obtain time-frequency analysis of signals, what which is not achievable using the Fourier transform itself. The transformation is by now the main concept of a vast theory named *time-frequency analysis* [4]. The basic idea in STFT is to bring time localization to the frequency analysis provided by the Fourier transform. This is done by first windowing the function to be analyzed and then taking the Fourier transform. Here, by "windowing the function" we mean to multiply it by translations of an appropriate function g, referred to as *window*, for example, a function of compact support. More precisely, given $f, g \in L^2(\mathbb{R}^n)$, the STFT of f with respect to g is the function

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$V_g f$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ via

$$V_g f(x, w) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t.w}dt.$$

It is straightforward that $V_g f(x, w) = (f.T_x\overline{g})^\wedge(w)$, where T_x is the operator of translation by x and $\wedge$ denotes the Fourier transform.

Interpreting the STFT in this way, one is able to prove results for the transform on the basis of the existing theorems of Fourier analysis. For example, Plancherel's theorem yields the very important *orthogonality relation for the STFT*, which asserts that

$$||V_g f||_2 = ||g||_2 \; ||f||_2. \tag{1.1}$$

Here $||V_g f||_2$ is computed with respect to the known product measure on $\mathbb{R}^n \times \mathbb{R}^n$.

It is easily understood that (1.1) is the STFT version of the isometry relation of Plancherel's theorem known as Plancherel's identity [3, Theorem 2.2.14]. Roughly speaking, the window g which is used to obtain the STFT from the Fourier transform manifests itself in the form $||g||_2$ in the variant (1.1) of Plancherel's identity.

In general, it is natural to seek for the STFT version of any result of Fourier analysis. One important result in Fourier analysis, which to the best of our knowledge has no STFT counterpart, is the Hausdorff–Young inequality stated below.

Theorem 1.1 (Hausdorff–Young inequality)**.** *Let $1 \leq q \leq 2$ and p be the conjugate exponent of q. Then the Fourier transform maps $L^q(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, and*

$$||\widehat{f}||_p \leq ||f||_q.$$

Our aim in this note is to prove an STFT version of the above theorem. This will be achieved in the next section. For now, we recall some preliminary results which can be found in [1].

Theorem 1.2 (Minkowski's inequality for integrals)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let f be a non-negative $(\mathcal{M} \bigotimes \mathcal{N})$-measurable function defined on $X \times Y$. Then for every $1 \leq p < \infty$,*

$$\left[\int_X \left(\int_Y f(x,y) \; d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int_Y \left[\int_X f(x,y)^p \; d\mu(x)\right]^{1/p} d\nu(y).$$

Proposition 1.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $0 < p < q < r \leq \infty$, then $L^p(\mu) \cap L^r(\mu) \subset L^q(\mu)$ and $||f||_q \leq ||f||_p^\lambda \; ||f||_r^{1-\lambda}$, where*

$\lambda \in (0,1)$ is given by

$$\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}.$$

2. The main result

The following is our main result, namely, *Hausdorff–Young inequality for the STFT*. In [5, Theorem 2.5] we proved a similar result for a directionally sensitive variant of the STFT known as *directional short-time Fourier transform* (DSTFT).

Theorem 2.1. *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then for every $2 < p < \infty$ with conjugate exponent q*

$$||V_g f||_p \leq ||g||_p \; ||f||_q. \tag{2.1}$$

Proof. Since $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$, $f.T_x\overline{g} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since $1 < q < 2$, this implies $f.T_x\overline{g} \in L^q(\mathbb{R}^n)$ for all such x, and also that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |V_g f(x,w)|^p dw\right)^{1/p} &= \left(\int_{\mathbb{R}^n} |(f.T_x\overline{g})^\wedge(w)|^p dw\right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^n} |f.T_x\overline{g}(v)|^q \, dv\right)^{1/q},\end{aligned}$$

in view of the Hausdorff–Young inequality (Theorem 1.1). Hence

$$\begin{aligned}||V_g f||_p^q &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |V_g f(x,w)|^p \, dw dx\right)^{q/p} \\ &\leq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f.T_x\overline{g}(v)|^q \, dv\right)^{p/q} dx\right)^{q/p}.\end{aligned}$$

Since $p/q > 1$, we may now apply Minkowski's inequality for integrals (Theorem 1.2) to the last integrals to obtain

$$\begin{aligned}||V_g f||_p^q &\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f.T_x\overline{g}(v)|^p dx\right)^{q/p} dv \\ &= \int_{\mathbb{R}^n} |f(v)|^q \left(\int_{\mathbb{R}^n} |g(v-x)|^p dx\right)^{q/p} \, dv.\end{aligned}$$

But

$$\int_{\mathbb{R}^n} |g(v-x)|^p dx = ||g||_p^p.$$

Hence

$$||V_g f||_p^q \leq (||g||_p \; ||f||_q)^q \,,$$

giving (2.1). $\square$

It is now appropriate to state some easy corollaries of the Hausdorff–Young inequality.

Corollary 2.2. *If g, p and q are as in Theorem 2.1, then the short-time Fourier transform V_g associated to g extends to a bounded operator of norm less than or equal to $||g||_p$ from $L^q(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. This follows from (2.1) and the density of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$. $\square$

Corollary 2.3. *With the assumptions of Theorem 2.1 one has*

$$||V_g f||_p \leq (||f||_2 \; ||g||_2)^{2/p} \; (||f||_1 \; ||g||_\infty)^{1-2/p} . \tag{2.2}$$

Proof. Since $g \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, by Proposition 1.3, $g \in L^p(\mathbb{R}^n)$ and

$$||g||_p \leq ||g||_2^\lambda \; ||g||_\infty^{1-\lambda}, \tag{2.3}$$

where $\lambda = 2/p$. Similarly, $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ implies

$$||f||_q \leq ||f||_1^\lambda \; ||f||_2^{1-\lambda} \tag{2.4}$$

with $\lambda = 2/q - 1 = 1 - 2/p$. Now, (2.2) follows from (2.3), (2.4) and (2.1). $\square$

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ISOMETRIES BETWEEN SPACES OF VECTOR-VALUED FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. Let X and Y be subsets of the real line with at least two points, and let E and F be strictly normed spaces. By $BV(X,E)$ we mean the space of E-valued functions of bounded variation on X with respect to the supremum norm $\|\cdot\|_\infty$. In this paper we give a complete description of a surjective, not assumed to be linear, isometry $T : BV(X,E) \longrightarrow BV(Y,F)$.

1. Introduction

There has been always considerable interest in studying isometries between different spaces of functions. In the context of spaces of continuous functions, the history dates back to the famous Banach-Stone theorem which characterizes complex-linear isometries of $C(X)$-spaces, where $C(X)$ denotes the space of all complex-valued continuous functions on a compact Hausdorff space X [3, 7]. Then Jerison obtained a vector analogue of the Banach-Stone theorem for $C(X,E)$-spaces as follows:

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Theorem 1.1 ([6]). *For compact Hausdorff spaces X and Y, a strictly convex Banach space E, and a surjective complex-linear isometry $T : C(X,E) \longrightarrow C(Y,E)$, there exist a homeomorphism $\varphi : Y \longrightarrow X$ and a continuous function J from Y into the space of bounded complex-linear operators on E equipped with the strong operator topology, such that $Tf(y) = J(y)(f(\varphi(y)))$ for all $y \in Y$ and $f \in C(X,E)$.*

This result has been extended in different directions. See [4] for a good survey of this topic. Here we focus on the study of isometries between spaces of vector-valued functions of bounded variation. First let us state some preliminaries.

Let X be a subset of the real line with at least two points and E be a normed space. It is said that a function $f : X \longrightarrow E$ has bounded variation if there exists a constant $k \geq 0$ such that

$$\sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\| \leq k,$$

for each $x_0, x_1, ..., x_n \in X, x_0 < x_1 < ... < x_n$ $(n \in \mathbb{N})$. The least such k is called the total variation of f and will be noted by $\mathcal{V}(f)$. We denote the space of all E-valued functions of bounded variation on X by $BV(X,E)$. We also set $BV(X) := BV(X,\mathbb{C})$. It should be noted that these functions are not necessarily continuous.

We also recall that a normed space E is called *strictly convex* if each $e \in S_E$ is an extreme point of the closed unit ball of E, where $S_E = \{e \in E : \|e\| = 1\}$.

Recently, Araujo [1] characterized surjective complex-linear isometries on $BV(X)$-spaces with respect to the complete norm $\|\cdot\|_\infty + \mathcal{V}(\cdot)$. Then, in [5], the author studied surjective real-linear isometries between spaces of functions of bounded variation with respect to the supremum norm and also the norm $\|\cdot\| := \max(\|\cdot\|_\infty, \mathcal{V}(\cdot))$. The aim of this paper is to characterize surjective, not assumed to be linear, isometries $T : BV(X,E) \longrightarrow BV(Y,F)$, where E and F are strictly convex normed spaces. More precisely, we give a weighted composition representation of $T - T0$ inducing a bijection between X and Y. Moreover, as a corollary, a vector-valued version of [5, Theorem 2.1] is obtained. We finally remark that our method has been motivated by ideas given in [2] and [5].

2. Main results

In the sequel, we assume that X and Y are compact Hausdorff spaces, E and F are strictly convex normed spaces. The object of this paper is to prove the following.

Theorem 2.1. *Let $T : BV(X,E) \longrightarrow BV(Y,F)$ be a surjective, not assumed to be linear, isometry with respect to the supremum norm. Then there exist a bijection $\psi : Y \longrightarrow X$, and a continuous function $J : Y \longrightarrow B(E,F)$ such that for all $f \in BV(X,E)$ and $y \in Y$,*

$$Tf(y) = T0(y) + J(y)(f(\psi(y))),$$

where $B(E,F)$ denotes the space of all continuous real-linear operators from E into F equipped with the strong operator topology.

The proof of the theorem will be given through several lemmas.

Lemma 2.2. *$S := T - T0$ is a surjective real-linear isometry.*

Given function $f : X \longrightarrow E$, we define the *cozero set* of f as $coz(f) = \{x \in X : f(x) \neq 0\}$.

Lemma 2.3. *S and S^{-1} are disjointness preserving, i.e., S and S^{-1} map functions with disjoint cozeros to functions with disjoint cozeros.*

Lemma 2.4. *Let $x_0 \in X$ and $e \in S_E$. There exists a unique $y_0 \in Y$ such that $coz(S(e\chi_{\{x_0\}})) = \{y_0\}$. Furthermore, $Sf(y_0) = 0$ for all $f \in BV(X,E)$ with $f(x_0) = 0$.*

We define the map $\varphi : X \longrightarrow Y$ by $\varphi(x) := y$, where y is the unique element of $coz(S(e\chi_{\{x\}}))$, obtained from the above lemma, for some $e \in S_E$.

Lemma 2.5. *The map $\varphi : X \longrightarrow Y$ is bijective.*

We define the function $J : Y \longrightarrow B(E,F)$ by $J(y)(e) = S\hat{e}(y)$ for all $e \in E$, where $\hat{e}$ denotes the function which is constantly e on X.

Now, putting $\psi := \varphi^{-1}$, it is not difficult to see that

$$Tf(y) = T0(y) + J(y)(f(\varphi(y))) \qquad (f \in BV(X,E)\ y \in Y),$$

which completes the proof of Theorem 2.1.

In scalar-valued case, we can easily obtain the following result which is a generalization of [5, Theorem 2.1].

Corollary 2.6. *Let $T : BV(X) \longrightarrow BV(Y)$ be a surjective (not necessarily linear) isometry. Then there exist a unimodular function Γ, a bijection $\psi : Y \longrightarrow X$, and a (possibly empty) subset K of Y such that for all $f \in BV(X)$ and $y \in Y$,*

$$Tf(y) = T0(y) + \Gamma(y) \begin{cases} \overline{f(\psi(y))} & y \in K \\ f(\psi(y)) & y \in Y \setminus K \end{cases}$$

where $\bar{\cdot}$ denotes the complex conjugate.

It should be noted that the following example shows that the induced bijection ψ need not be a homeomorphism (compare with Theorem 1.1).

Example 2.7 ([5])**.** Consider $X = (0, 1] \cup \{-1\}$, $Y = [0, 1) \cup \{2\}$, and define the complex-linear isometry $T : BV(X) \longrightarrow BV(Y)$ by

$$Tf(y) = \begin{cases} f(y) & y \in (0,1) \\ f(-1) & y = 0 \\ f(1) & y = 2. \end{cases}$$

Then $Tf = f \circ \psi$, where the bijection ψ ($\psi(y) = y$ if $y \in (0, 1)$, $\psi(2) = 1$, and $\psi(0) = -1$) is not a homeomorphism.

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ON SOME WEIGHTED TRANSLATION OPERATORS ON $L^2(\Sigma)$

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Abstract. In this note we give some measure-theoretic characterizations of classical properties of weighted composition operators and their products on $L^2(\Sigma)$.

1. Introduction

Let (X, Σ, μ) be a complete sigma finite measure space and let $\mathcal{A}$ be a sub-sigma finite algebra of Σ. We use the notation $L^p(\mathcal{A})$ for $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ and henceforth we write μ in place $\mu_{|\mathcal{A}}$. If $B \subset X$, let $\mathcal{A}_B$ denote the relative completion of the sigma-algebra generated by $\{B \cap A : A \in \mathcal{A}\}$ and denote the complement of B by B^c. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear spaces of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with

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* Speaker.

respect to μ, that is, φ is non-singular. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite.

The symbol $E^{\varphi^{-1}(\Sigma)}$ represents the conditional expectation operator with respect to $\varphi^{-1}(\Sigma)$. As an operator on $L^2(\Sigma)$, it is the orthogonal projection onto $L^2(\varphi^{-1}(\Sigma))$ but $E^{\varphi^{-1}(\Sigma)}f$ is also defined for f in any $L^p(\Sigma)$, $1 \leq p \leq \infty$, as well as for any $f \geq 0$ [2]. If there is no possibility of confusion, we write Ef in place of $E^{\varphi^{-1}(\Sigma)}f$. For further discussion of the conditional expectation operator see [4]. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator W on $L^p(\Sigma)$ with $1 \leq p \leq \infty$ induced by non-singular measurable φ and u is given by $W = M_u \circ C_\varphi$ where M_u is a multiplication operator and C_φ is a composition operator on $L^p(\Sigma)$ defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively. It is a classical fact that $W \in \mathcal{B}(L^p(\Sigma))$, with $1 \leq p \leq \infty$, if and only if $J := hE(\|u\|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$ and $W \in \mathcal{B}(L^\infty(\Sigma))$ if and only if $u \in L^\infty(\Sigma)$ [2]. A result of Hoover, Lambert and Quinn [2] shows that the adjoint W^* of W on $L^2(\Sigma)$ is given by $W^*(f) = hE^{\varphi^{-1}(\Sigma)}(\bar{u}f) \circ \varphi^{-1}$. From this it follows that $W^*W = M_J$ and $WW^*(f) = M_{u(h\circ\varphi)}E(\bar{u}f)$.

Products of operators appear more often in the service of the study of other operators. More precisely, for any operator T, there exists a decomposition $T = (U + K)S$, where U is a partial isometry, K is a compact operator and S is a strongly irreducible operator [5]. Weighted composition operators and their products have been used to provide examples and illustrations of many operator theoretic properties. In several cases major conjectures in operator theory have been reduced to the weighted composition operators. The purpose of this note is to find some new characterizations of weighted composition operators on $L^2(\Sigma)$ and present a relationship between uC_φ and their products.

2. Main results

Lemma 2.1. *For nonsingular measurable transformations φ_1 and φ_2, let $\varphi_3^{-1}(\Sigma)$ be a sub-sigma finite algebra of Σ and $C_{\varphi_i}, W_i \in \mathcal{B}(L^2(\Sigma))$. Then the following assertions hold*

(a) *If $C_{\varphi_3} = C_{\varphi_2}C_{\varphi_1} \in \mathcal{B}(L^2(\Sigma))$, then $C^*_{\varphi_3}(f) = h_1E_1(h_2E_2(f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}$.*

(b) *$h_3 = h_1E_1(h_2) \circ \varphi_1^{-1}$.*

(c) $\dfrac{h_1 \circ \varphi_3}{h_3 \circ \varphi_3} = \dfrac{1}{E_1(h_2) \circ \varphi_2}$.

Theorem 2.2. *Let $\varphi_3^{-1}(\Sigma)$ be a sub-sigma finite algebra of Σ, $h_i \in L^\infty(\Sigma)$ and $W_i \in \mathcal{B}(L^2(\Sigma))$. Then the following assertions hold.*

(a) $W_3 = u_3 C_{\varphi_3} = u_2(u_1 \circ \varphi_2) C_{\varphi_2} \circ C_{\varphi_1} \in \mathcal{B}(L^2(\Sigma))$ *if and only if* $J_3 := h_1 E_1(|u_1| J_2) \circ \varphi_1^{-1} \in L^\infty(\Sigma)$ *and in this case* $\|W_3\| = \|h_1 E_1(|u_1| J_2) \circ \varphi_1^{-1}\|_\infty^{1/2}$.

(b) W_3 *is injective if and only if* $\sigma(h_1) = \sigma(E_1(|u_1|^2 J_2)) = X$.

Lemma 2.3. *Let $W_i \in \mathcal{B}(L^2(\Sigma))$, $\varphi_3 = \varphi_1 \circ \varphi_2$ and $u_3 = u_2(u_1 \circ \varphi_2)$ Then the following assertions hold.*

(a) $J_3 \circ \varphi_3 = (h_1 \circ \varphi_3) E_1(|u_1|^2 J_2) \circ \varphi_2$.

(b) $W_3^*(f) = h_1 E_1(\bar{u_1} h_2 E_2(\bar{u_2} f) \circ \varphi_2^{-1}) \circ \varphi_1^{-1}$.

(c) $W_3^* W_3(f) = (h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}) f$.

(d) $W_3 W_3^*(f) = u_3 (h_1 \circ \varphi_3) E_1(\bar{u_1} h_2 E_2(\bar{u_2} f) \circ \varphi_2^{-1}) \circ \varphi_2$.

(e) $W_3^* W_3 W_3(f) = (u_3 h_1 E_1(|u_1|^2 J_2) \circ \varphi_1^{-1}) f \circ \varphi_3$.

(f) $W_3 W_3^* W_3(f) = (u_3 (h_1 \circ \varphi_3) E_1(|u_1|^2 J_2) \circ \varphi_2) f \circ \varphi_3$.

(g) $W_3 W_3^*(f) = u_3 (h_3 \circ \varphi_3) E_3(\bar{u_3} f)$.

Lemma 2.4. *Let $uC_\varphi \in \mathcal{B}(L^2(\Sigma))$. Then the following assertions hold.*

(a) *([1]) uC_φ is normal if and only if $\left(\varphi^{-1}(\Sigma)\right)_{\sigma(u)} = \Sigma_{\sigma(u)}$ and $J = \chi_{\sigma(u)} J \circ \varphi$.*

(b) *([1]) uC_φ is quasinormal if and only if $J = J \circ \varphi$ on $\sigma(u)$.*

(c) *([3]) uC_φ is hyponormal if and only if $\sigma(u) \subseteq \sigma(J)$ and $(h \circ \varphi) E(\frac{|u|^2}{J}) \leq \chi_{\sigma(E(u))}$ a.e.*

Theorem 2.5. *Let $u_i C_{\varphi_i} \in \mathcal{B}(L^2(\Sigma))$ with $J_1 \circ \varphi_2 = J_1$ and $J_2 \circ \varphi_1 = J_2$ on $\sigma(u_1)$ and $\sigma(u_2)$, respectively. Then the following holds.*

(a) *If $u_1 C_{\varphi_1}$ and $u_2 C_{\varphi_2}$ are normal (quasinormal), then $u_3 C_{\varphi_3}$ is a normal (quasinormal) operator.*

(b) *If $u_1 C_{\varphi_1}$ and $u_2 C_{\varphi_2}$ are hyponormal and $h_2 E_2(\frac{|u_2|^2}{J_2}) \circ \varphi_2^{-1}$ is a $\varphi_1^{-1}(\Sigma)$-measurable function, then $u_3 C_{\varphi_3}$ is a hyponormal operator.*

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CONDITIONAL TYPE TOEPLITZ OPERATORS ON $L^2_a(\mathbb{D})$

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Abstract. In this paper, we initiate the study of a new class of conditional type operators, which we call conditional type Toeplitz operators. Sufficient conditions for boundedness and compactness of conditional type Toeplitz operators on the Bergman space $L^2_a(\mathbb{D})$ will be presented. Also, some differences between conditional type Toeplitz operators and Toeplitz operators will be illustrated by examples.

1. Introduction

Let (X, Σ, μ) be a probability measure space and let $\mathcal{A}$ be a sub-algebra of Σ. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. The collection of $\mathcal{A}$-measurable complex-valued functions on X will be denoted by $L^0(\mathcal{A})$. We take $L^2(\mathcal{A}) = L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$. For each non-negative function $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there

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* Speaker.

exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu,$$

where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$, uniquely defined by the assignment $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is a contractive orthogonal projection onto $L^2(\mathcal{A})$. For fixed $\mathcal{A} \subseteq \Sigma$, set $E^{\mathcal{A}} = E$. The domain of E contains $L^1(\Sigma) \cup L^0_+(\Sigma)$, where $L^0_+(\Sigma) = \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on conditional expectation see [3].

We now restrict our attention to the case $(\mathbb{D}, \mathcal{M}, A)$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets in $\mathbb{D}$ and A is the normalized area measure in $\mathbb{D}$. For $1 \leq p < \infty$, the Bergman space $L^p_a(\mathbb{D}) = L^p_a(\mathcal{M})$ is a closed subspace of $L^p(\mathcal{M})$ consisting of analytic functions. Let P be the Bergman projection. For $u \in L^\infty(\mathcal{M})$, the operator T_u defined on $L^2_a(\mathbb{D})$ by $T_u f = P(uf)$ is called Toeplitz operator. When $u \in H^\infty(\mathbb{D})$, the space of bounded analytic functions on $\mathbb{D}$, then T_u is reduced to the multiplication operator M_u. For general information on this topic one can refer to excellent monograph [4].

The operator $T = EM_u$ have been defined as combination of multiplication operator and conditional expectation operator. Lambert and others researcher have obtained many property of T such as boundedness, compactness, spectrum and so on. For more details about this type of operators one can refer to [1, 2].

In this paper, we introduce the concept of conditional type Toeplitz operators, PT, on the Bergman space $L^2_a(\mathbb{D})$ and present some algebraic and analytic properties of these types of operators. In Examples 2.6, 2.10 and 2.12 we see that conditional type Toeplitz operators with same properties for u, have different behavior relative to Toepliz operators. In particular, a sufficient condition for boundedness and compactness of mentioned operators will be presented

2. Main results

Suppose $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets in $\mathbb{D}$ and $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ and $E = E^{\mathcal{A}}$ is the related conditional expectation operator.

Definition 2.1. For $u \in L^\infty(\mathcal{M})$, the conditional type Toeplitz operator induced by the pair (u, E) is denoted by T^E_u and defined as follows:

$$T^E_u = PEM_u : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D}), \qquad f \to PE(uf),$$

where M_u is the multiplication operator. Note that, since $uf \in L^2(\mathcal{M}) \subseteq \mathcal{D}(E)$ and $E(uf) \in L^2(\mathcal{A}) \subseteq L^2(\mathcal{M})$, so the linear operator T_u^E is well defined.

Let $\mathfrak{T}_2 = \{u \in L^2(\mathcal{M}) : uL_a^2(\mathbb{D}) \subseteq L^2(\mathcal{M})\}$. Note that $L^\infty(\mathcal{M}) \subseteq \mathfrak{T}_2 \subseteq L^1(\mathcal{M})$ and that $\mathfrak{T}_2$ is a vector space. For $u \in \mathfrak{T}_2$, let $\mathcal{T}_u^E$ be the corresponding conditional type Toeplitz operator. For $u \in L^\infty(\mathcal{M})$, $\mathcal{T}_u^E = T_u^E$. So $\mathcal{T}_u^E$ is a generalization of T_u^E.

Lemma 2.2. *Let $u \in L^2(\mathcal{M})$. Then the operator $EM_u : L^2(\mathcal{M}) \to L^2(\mathcal{A})$ is bounded if and only if $E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|EM_u\| = \|\sqrt{E(|u|^2)}\|_\infty$.*

Proposition 2.3. *If $E(|u|^2) \in L^\infty(\mathcal{A})$, then $\mathcal{T}_u^E$ is a bounded linear operator on $L_a^2(\mathbb{D})$.*

If $\mathcal{M} = \mathcal{A}$, then $E = I$, the identity operator. In this case $\mathcal{T}_u^E = \mathcal{T}_u$

Example 2.4. Let $\mathcal{A} = \{\emptyset, \mathbb{D}\}$. Then $E(f)(z) = \int_{\mathbb{D}} f(w)dA(w)$, and so

$$\mathcal{T}_u^E(f)(z) = P(E(uf))(z) = \int_{\mathbb{D}} u(w)f(w)dA(w).$$

Proposition 2.5. *Suppose a and b are complex numbers and u and v are in $\mathfrak{T}_2$ such that the conditional type Toeplitz operators induced by them are bounded. Then*

(i) $\mathcal{T}_{au+bv}^E = a\mathcal{T}_u^E + bT_v^E$ *and* $(\mathcal{T}_u^E)^* = PM_{\bar{u}}E$.
(ii) *If u be a $\mathcal{A}$-measurable and $u \geq 0$, then $\mathcal{T}_u^E \geq 0$.*

For classical Toeplitz operator $\mathcal{T}_u = PM_u$ on $L_a^2(\mathbb{D})$, $\mathcal{T}_u \equiv 0$ if and only if $u \equiv 0$. But the analogous fact does not hold for conditional type Toeplitz operators, in general.

Example 2.6. Suppose again that $\mathcal{A} = \{\emptyset, \mathbb{D}\}$ and u is a nonzero analytic function on $\mathbb{D}$ such that $u(0) = 0$. In this case we have, $\mathcal{T}_u^E \equiv 0$ on $L_a^2(\mathbb{D})$, but $u \not\equiv 0$.

Proposition 2.7. *Suppose u is an $\mathcal{A}$-measurable function on $\mathbb{D}$. Then $\mathcal{T}_u^E \equiv 0$ implies that $u \equiv 0$ if and only if the linear combinations of $\{\overline{E(z^i)}E(z^j)\}_{i,j=0}^\infty$ is dense in $L^2(\mathcal{A})$.*

Corollary 2.8. *Suppose u is an $\mathcal{A}$-measurable function on $\mathbb{D}$ and M denotes the linear combinations of $\{\overline{E(z^i)}E(z^j)\}_{i,j=0}^\infty$. Then the following assertions hold.*

(i) *$\mathcal{T}_u^E$ is self-adjoint if and only if $u - \bar{u}$ is perpendicular to $\overline{M}$.*
(ii) *If $\overline{M} = L^2(\mathcal{A})$, then $\mathcal{T}_u^E$ is self-adjoint if and only if $u = \bar{u}$.*

Theorem 2.9. *Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{M}$, $(\mathbb{D}, \mathcal{A}, A_{|\mathcal{A}})$ can be partitioned as $\mathbb{D} = (\cup_{n\in\mathbb{N}} C_n) \bigcup B$ and $T = \mathcal{T}_u^E$ is bounded on $L_a^2(\mathbb{D})$. If $u(B) = 0$ ($u(z) = 0$ for all $z \in B$) and for any $\epsilon > 0$, $A(C_n \cap G^\epsilon(u)) > 0$ for finitely many n, where $G^\epsilon(u) = \{z \in \mathbb{D} : E(|u|)(z) \geq \epsilon\}$, then $\mathcal{T}_u^E$ is compact.*

Example 2.10. Suppose $u \in L^1(\mathcal{M})$ is harmonic. Then $\mathcal{T}_u$ is compact if and only if $u \equiv 0$ (see [4]). Let $\mathcal{A} = \{\emptyset, \mathbb{D}\}$ and $u \in H^\infty(\mathbb{D})$, in this case $\mathcal{T}_{\bar{u}}^E$ is compact while $\mathcal{T}_{\bar{u}}$ is not compact.

Theorem 2.11. *Suppose $u \in \mathfrak{T}_2$. If for $n \geq 0$, Eu and $|z|^{2n}\bar{u}$ are perpendicular to $zL_a^2(\mathbb{D})$ and for $n \geq 0$, $|z|^{2n}u$ is perpendicular to $L_a^2(\mathbb{D})$ then $\mathcal{T}_u^E$ is a diagonal operator on $L_a^2(\mathbb{D})$.*

When $u \in L^2(\mathcal{M})$ is not radial then $\mathcal{T}_u$ is not diagonal on $L_a^2(\mathbb{D})$. In spite of classical Toeplitz operator cases, there are functions $u \in L^2(\mathcal{M})$ such that u is not radial but the induced conditional type Toeplitz operator $\mathcal{T}_u^E$ is diagonal.

Example 2.12. Let $u(z) = p_n(z)$, such that $p_n(0) \neq 0$. Suppose E and $\mathcal{A}$ are as in Example 2.4. Since $\mathcal{T}_u^E(z^k)(z) = \int_{\mathbb{D}} u(w)w^k dA(w)$, using mean value property for harmonic functions, we have

$$\mathcal{T}_u^E(z^k) = \begin{cases} p_n(0) & k = 0 \\ 0 & k \geq 1, \end{cases}$$

thus $\mathcal{T}_u^E(z^k) = c_k z^k$, where $c_0 = p_n(0)$ and $c_k = 0$ for $k \geq 1$, so $\mathcal{T}_u^E$ is diagonal while $\mathcal{T}_u$ is not diagonal.

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MIKUSIŃSKI OPERATIONAL CALCULUS WITH CONVOLUTION QUADRATURE FOR FRACTIONAL SUBDIFFUSION EQUATION

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Abstract. The convolution quadrature can be well described by the Mikusiński operational calculus using convolution semigroup. We present a numerical method for the anomalous diffusion equation by using convolution quadrature for temporal discretization and Bernstein polynomial projection for spatial representation. The stability and accuracy of the method are then discussed.

1. Introduction

The Mikusinski operational calculus entities may be regarded as operators or distributions [1].

Definition 1.1. Let $a, b \in \mathcal{C}$, the space of continuous functions of a nonnegative variable, and let $a + b$ and αa, $(\alpha \in \mathbb{R})$, be the functions whose values at t are defined as $a(t) + b(t)$ and $\alpha a(t)$, respectively, and

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define ab as

$$ab(t) = \int_0^t a(u)b(t-u)du. \tag{1.1}$$

Let h be the restricted Heaviside function on $t \geq 0$. Then, $h^n(t) = t^{n-1}/(n-1)!$, $n \in \mathbb{N}$. More generally, we define tentatively

$$h^\alpha(t) = t^{\alpha-1}/\Gamma(\alpha) \qquad (\Re\alpha > 0). \tag{1.2}$$

Clearly $h \in \mathcal{C}$ and $hf(t) = \int_0^t f(u)du$, i.e., h acts as the integration operator and h^n for the n-times integration. The operator h^α is in fact the Riemann-Liouville fractional integration of order α.

The set $\mathcal{C}$ is a vector space and a commutative ring without any unit element, called the convolution ring.

Theorem 1.2 ([1])**.** *Let $a, b \in \mathcal{C}$. If $ab = 0$, then $a = 0$ or $b = 0$, i.e., the ring has no divisors of zero.*

Therefore, the ring can be extended to a field, the so-called field of convolution quotients, $\mathcal{F}$. It is proved that it is also an algebra (see Section 2.4 of [1] for more details).

The convolution quadrature provides an approximation for

$$\int_0^t k(t-u)f(u)du \qquad (t > 0), \tag{1.3}$$

at nodes $t = t_n = nh$ for $n = 1, 2, \ldots$, with a step size $h > 0$ with a discrete convolution given by

$$W_n^T F_n = \sum_{j=0}^{n} \bar{w}_{n-j} f_j \qquad (n = 1, \ldots, N), \tag{1.4}$$

where $W_n^T = [\bar{w}_{n-j}, j = 0, ..., n], F_n = [f_j = f(jh), j = 0, ..., n]$ and the weights are computed by

$$K(\frac{\delta(\xi)}{h}) = \sum_{j=0}^{\infty} w_j \xi^j, \tag{1.5}$$

$$\bar{w}_n = \frac{w_n}{2}, \qquad \bar{w}_j = w_j, \qquad j \neq n,$$

in which K is the Laplace transform of k and $\delta(\xi) = (1-\xi) + (1-\xi)^2/2$ is based on the second order backward difference formula (BDF);[5]. As the following theorem shows the method with second order BDF is convergent (see, e.g., [5]).

Theorem 1.3. *Let K be an analytic and bounded function, $|K(s)| \leq \bar{M}|s-\sigma|^{-\mu}$ for $|\arg(s-\sigma)| < \phi$ with $\phi > \pi/2$ for some real $\mu > 0, \bar{M}$*

and σ, and let $f(t) = ct^\gamma$ for a constant c and $\gamma \geq 0$. Then, for the error of (1.4) we have

$$\left| \int_0^t k(t-u) f(u) du - \sum_{j=0}^{n} \bar{w}_{n-j} f_j \right| = \begin{cases} O(h^{1+\gamma}) & 0 \leq \gamma < 1 \\ O(h^2) & \gamma \geq 1 \end{cases}$$

In this paper, we use convolution quadrature discretization along with the Bernstein polynomials for developing a numerical method to solve the following fractional diffusion equation,

$$D_t^\alpha u\left(x,y,t\right) = \kappa \Delta u\left(x,y,t\right) + S\left(x,y,t\right), \qquad \left(x,y,t\right) \in \Omega \times (0,T] \tag{1.6}$$

with initial and boundary conditions

$$u\left(x,y,0\right) = g\left(x,y\right), \qquad \left(x,y\right) \in \Omega, \tag{1.7}$$

$$u\left(x,y,t\right) = 0, \qquad \left(x,y,t\right) \in \partial\Omega \times (0,T]. \tag{1.8}$$

Here, $D_t^\alpha u$ is the Caputo derivative of order α, $0 < \alpha < 1$, defined as

$$D_t^\alpha u\left(x,t\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{1}{\left(t-s\right)^\alpha} \frac{\partial u\left(x,s\right)}{\partial s} ds. \tag{1.9}$$

2. Main results

For $\alpha \in (0,1)$, due to the relation between Caputo and Riemann-Liouville derivative, $D_t^\alpha u =^{RL} D_t^\alpha (u-g)$ with $g = u^0$, the problem is discretized at $t = t_{k+1} = (k+1)\tau$ where $\tau = T/M$ and M is the number of nodes for $0 < t \leq T$, as

$$\sum_{j=0}^{k+1} \omega_{k+1-j} 1_\tau u^j = \kappa \Delta u^{k+1} + S^{k+1} + \bar{\partial}_\tau^\alpha u^0,$$

i.e., as the following semi-discrete scheme

$$\omega_0 u^{k+1} - \kappa \Delta u^{k+1} = S^{k+1} + \bar{\partial}_\tau^\alpha u^0 - \sum_{j=0}^{k} \omega_{k+1-j} 1_\tau u^j. \tag{2.1}$$

The Galerkin formulation is to find $u_N^{k+1} \in (\mathbb{P}_N^0)^2$ such that for all $v_N \in (\mathbb{P}_N^0)^2$:

$$(w_0 u_N^{k+1}, v_N) + \kappa(\nabla u_N^{k+1}, \nabla v_N) \tag{2.2}$$

$$= (-w_1 u_N^k - \sum_{j=1}^{k-1} w_j u_N^{k-j} + w_{k+1} u_N^0 + S^{k+1}, v_N), \tag{2.3}$$

with (f,g) being the standard L^2-norm; [3, 2, 4]. We use the Bernstein basis as the trial and its dual as the test functions in the above scheme

Table 1. Temporal convergence at $t = 1$ for Example 2.1.

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
M	L^∞	rate	L^∞	rate	L^∞	rate
10	1.33E-03		1.87E-03		1.52E-03	
20	5.72E-04	1.22	7.03E-04	1.41	4.71E-04	1.69
40	2.58E-04	1.15	2.74E-04	1.36	1.46E-04	1.69
80	1.19E-04	1.12	1.07E-04	1.35	4.48E-05	1.70

to obtain the matrix formulation of the method. It is discussed that the matrices have special interesting banded structures [3, 2, 4].

Example 2.1. Now consider the problem (1.6)-(1.8) with $\kappa = 1$, the initial condition $g = xy(1-x)(1-y) \in \dot{H}^2(\Omega)$ and no source term, $S = 0$.

$$u(x,y,t) = \sum_{n,m=1}^{\infty} b_{m,n} \sin n\pi x \sin m\pi xy E_{\alpha,1}(-(n^2+m^2)\pi^2 t^\alpha),$$

$$b_{m,n} = \frac{16(1-(-1)^n)(1-(-1)^m)}{m^3 n^3 \pi^6}.$$

Table 1 reports the error and rate of convergence for the method.

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APPROXIMATION OF CONTINUOUS FUNCTIONS VANISHING AT INFINITY BY WEIGHTED TRANSFORM

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ABSTRACT. In this paper, we introduce weighted and inverse weighted transforms with respect to a new partition on unbounded interval $(-\infty, \infty)$. We show that real valued continuous functions vanishing at infinity defined on unbounded interval $(-\infty, \infty)$ can be approximated by the inverse weighted transform with an arbitrary precision.

1. INTRODUCTION

Approximation theory has many applications in various areas such as functional analysis, the theory of polynomials approximation and numerical solutions of differential integral equations. In many problems because of the complexity of computation, functions can be approximated with methods that convert continuous functions into n-dimensional vector such that computing is easier. One of these methods is *fuzzy transform* which introduced by Perfilieva in 2006 [3].

Contiuning the study of accuracy of approximation of continuous functions on $[a, b]$ $(f \in C_c([a, b]))$ by fuzzy transform [3] and F_φ-transform on a bounded subset [2] and weighted transform on unbouded set [1], in this paper we construct an approximation model for continuous functions vanishing at infinity on unbounded interval $(-\infty, \infty)$ by introducing the weighted and inverse weighted transforms. Finally, an example that shows the validity of our results will be given.

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* Speaker.

2. Basic concepts

Throughout the paper, let $X = (-\infty, +\infty)$. Denote by $L_0^1(X)$ the set of all $f \in C_0(X)$ with $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Definition 2.1. Let $\psi \in L_0^1(X)$ be a function from X into $(0, 1]$ and $\{x_0, x_1, ..., x_n, x_{n+1}\}$ be a subset of X with $x_0 = x_1 < ... < x_n = x_{n+1}$. The set $\mathcal{B} = \{B_1, ..., B_n\}$ is called a *ψ-partition* of X if each B_k is a continuous map from X into $[0, 1]$ and fulfills the following conditions for $k = 1, ..., n$:

(i) $B_k(x_k) = \psi(x_k)$ for $k = 1, ..., n$.
(ii) $B_1(x) = 0$ for $x \notin (-\infty, x_2)$.
(iii) $B_1(x) = \psi(x)$ for $x \in (-\infty, x_1]$.
(iv) $B_k(x) = 0$ for $x \notin (x_{k-1}, x_{k+1})$, $k = 2, ..., n-1$.
(v) $B_n(x) = 0$ for $x \notin (x_{n-1}, \infty)$.
(vi) $B_n(x) = \psi(x)$ for $x \in [x_n, \infty)$.
(vii) $\sum_{k=1}^{n} B_k(x) = \psi(x)$ for $x \in X$.

We can construct a ψ-partition of X by defining B_k, $k = 1, ..., n$, as follows: for $k = 2, ...n-1$ define

$$B_1(x) = \begin{cases} \psi(x) & x \in (-\infty, x_1] \\ \psi(x)\left(\frac{x-x_2}{x_1-x_2}\right) & x \in [x_1, x_2] \\ 0 & \text{otherwise;} \end{cases}$$

$$B_k(x) = \begin{cases} \psi(x) - B_{k-1}(x) & x \in [x_{k-1}, x_k] \\ \psi(x)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right) & x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise;} \end{cases}$$

and

$$B_n(x) = \begin{cases} \psi(x) - B_{n-1}(x) & x \in [x_{n-1}, x_n] \\ \psi(x) & x \in [x_n, \infty) \\ 0 & \text{otherwise;} \end{cases}$$

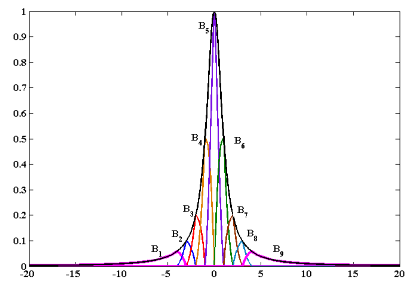

FIGURE 1. A ψ-partition on X with $\psi(x) = \frac{1}{1+x^2}$.

Obviously, $\mathcal{B} = \{B_1, ..., B_n\}$ is a ψ-partition of X. Figure 1 shows the ψ-partition for X with $\psi(x) = 1/(1 + x^2)$ and $n = 9$.

Definition 2.2. Let $\psi \in L_0^1(X)$, $\mathcal{B} = \{B_1, ..., B_n\}$ be a ψ-partition of X and $f \in C_0(X)$. The linear map $\mathcal{F}_\psi : C_0(X) \to \mathbb{R}^n$ defined by n-tuple of real numbers $\mathcal{F}_\psi[f] = [F_1, ..., F_n]$, where

$$F_k = \frac{\int_{-\infty}^{+\infty} f(x)\, B_k(x)\, dx}{\int_{-\infty}^{+\infty} B_k(x)\, dx},$$

for $k = 1, ..., n$, is called the *weighted transform* of f with respect to ψ-partition $\mathcal{B} = \{B_1, ..., B_n\}$.

In the following definition, we introduce the inverse weighted transform for $f \in C_0(X)$.

Definition 2.3. Let $\psi \in L_0^1(X)$ and $\mathcal{B} = \{B_1, ..., B_n\}$ be a ψ-partition of X. If $f \in C_0(X)$ and $\mathcal{F}_\psi[f] = [F_1, ..., F_n]$ is the weighted transform of f with respect to ψ-partition $\mathcal{B}$, then the *inverse weighted transform* based on ψ-partition $\mathcal{B} = \{B_1, ..., B_n\}$ is defined by function $\mathcal{T}_n^F(x) = \sum_{k=1}^{n} F_k \, \frac{B_k(x)}{\varphi(x)}$.

3. MAIN RESULTS

In the sequel, it is shown that every $f \in C_0(X)$ can be approximated on $(-\infty, x_1]$ by F_1, on $[x_k, x_{k+1}]$, for all $k = 1, ..., n-1$, by F_k and finally on $[x_n, \infty)$ by F_n.

Theorem 3.1. *Let $\psi \in L_0^1(X)$ and $f \in C_0(X)$. Then for any $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ and a ψ-partition $\mathcal{B} = \{B_1, ..., B_{n_\varepsilon}\}$ such that*

(i) $|f(t) - F_1| < \varepsilon$ *for* $t \in (-\infty, x_1]$.
(ii) $|f(t) - F_i| < \varepsilon$ *for* $t \in [x_k, x_{k+1}]$, $k = 1, ..., n-1$ *and* $i = k, k+1$.
(iii) $|f(t) - F_n| < \varepsilon$ *for* $t \in [x_n, \infty)$.

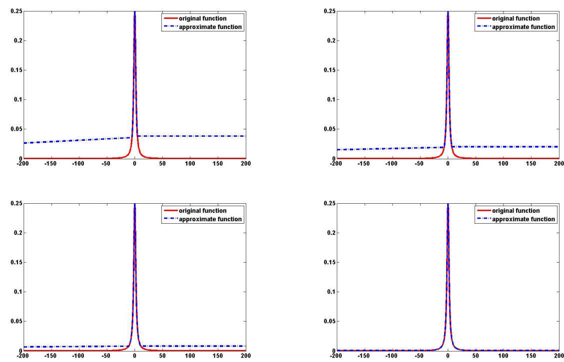

FIGURE 2. The approximation of f by the inverse weighted transform. Top left:$n = 120$. Top right:$n = 180$. Bottom left:$n = 300$. Bottom right:$n = 900$.

(iv) $|F_k - F_{k+1}| < \varepsilon$ *for* $k = 1, ..., n - 1$.

Now, we show that the inverse weighted transform $\mathcal{T}_n^F$ can approximate the original function f.

Theorem 3.2. *Let* $\psi \in L_0^1(X)$ *and* $f \in C_0(X)$. *Then, for any* $\varepsilon > 0$ *there exist* $n_\varepsilon \in \mathbb{N}$ *and a* ψ*-partition* $\mathcal{B} = \{B_1, ..., B_{n_\varepsilon}\}$ *such that*

$$\|f - \mathcal{T}_{n_\varepsilon}^F\|_u < \varepsilon,$$

where $\|f\|_u = \sup_{x\in X} |f(x)|$.

Corollary 3.3. *Let* $\psi \in L_0^1(X)$ *and* $f \in C_0(X)$. *Then there exists a sequence of inverse weighted transforms that uniformly converges to* f.

The following example supports accuracy of Theorem 3.2.

Example 3.4. Let f and ψ be defined on $X = (-\infty, +\infty)$ by

$$f(x) = \frac{1}{x^2 + 4}, \qquad \psi(x) = \frac{1}{\cosh(x)}.$$

Clearly, $f \in C_0(X)$ and $\psi \in L_0^1(X)$. Figure 2 shows the desired approximation of the original function f with respect to ψ-partition B_k's, $k = 1, ..., n$, which has been defined as in Theorem 3.1. In our figures, the original function has been marked with solid line and the approximation function has been marked with dashed line.

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ON CERTAIN PROBABILITY SPACES

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Abstract. We introduce a certain class of probability spaces and present some examples of this class. Also we prove some basic results and give a characterization of this class of probability spaces.

1. Introduction and preliminaries

In statistical calculations and some inequalities we need the relation $\sum_{n=1}^{\infty} E(\xi_n^2) < \infty$ holds whenever $\sum_{n=1}^{\infty} n^q E(|\xi_n|) < \infty$ for some $q \in [1,\infty)$, where $\{\xi_n\}_n$ is a sequence of random variables and $E(\xi_n^2)$ and $E(|\xi_n|)$ are the expectations of integrable random variables ξ_n^2 and $|\xi_n|$, respectively. Motivated by this, here we introduce and study a certain class of probability spaces enjoy such a property.

Let (Ω, F, P) be a probability space. Then a random variable $\xi : \Omega \longrightarrow \mathbb{R}$ is said to be integrable if $\int_\Omega |\xi| dP < \infty$. The family of integrable random variables $\xi : \Omega \longrightarrow \mathbb{R}$ will be denoted by $L^1(\Omega, F, P)$ or simply L^1. A random variable $\xi : \Omega \longrightarrow \mathbb{R}$ is called square integrable if $\int_\Omega |\xi|^2 dP < \infty$. The family of square integrable random variables $\xi : \Omega \longrightarrow \mathbb{R}$ will be denoted by $L^2(\Omega, F, P)$ or L^2.

The expectation of an integrable random variable ξ is defined by $E(\xi) = \int_\Omega \xi dP$. Also the variance of a square integrable random

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variable ξ is defined by $\operatorname{var}(\xi) = E(\xi^2) - E(\xi)^2$. Clearly, $L^2 \subseteq L^1$; but the converse is not the case in general. Indeed let $\Omega = (0,1]$ with the σ-field $F = B((0,1])$ of all Borel subsets $B \subseteq (0,1]$, and Lebesgue measure $P = Leb$ on $(0,1]$. Then $\xi : \Omega \longrightarrow \mathbb{R}$ defined by $\xi(w) = 1/\sqrt{w}$ is a random variable such that $\xi \in L^1$ but $\xi \notin L^2$. So $L^2((0,1], B((0,1]), P) \subsetneqq L^1((0,1], B((0,1]), P)$.

Let Ω be a non-empty set and $\omega_0 \in \Omega$. Then $(\Omega, 2^\Omega, \delta_{\omega_0})$ is a probability space, where $\delta_{\omega_0} : 2^\Omega \longrightarrow \mathbb{R}$ is the Dirac measure at ω_0 defined by

$$\delta_{\omega_0}(A) = \begin{cases} 1 & \omega_0 \in A \\ 0 & \omega_0 \notin A \end{cases}$$

2. Main Results

Definition 2.1. Let (Ω, F, P) be a probability space and $q \geq 1$. Then (Ω, F, P) is a q-nice probability space if for each sequence $\{\xi_n\}_n$ of random variables on (Ω, F, P), $\sum_{n=1}^{\infty} E(\xi_n^2) < \infty$ whenever we have $\sum_{n=1}^{\infty} n^q E(|\xi_n|) < \infty$.

Example 2.2. (1) Let Ω be a non-empty set and $\omega_0 \in \Omega$. Then $(\Omega, 2^\Omega, \delta_{\omega_0})$ is a q-nice probability space for all $q \geq 1$. Note that if ξ is a random variable on $(\Omega, 2^\Omega, \delta_{\omega_0})$. Then it is obvious that $E(\xi) = \xi(\omega_0)$. So for each sequence $\{\xi_n\}_n$ of random variables on $(\Omega, 2^\Omega, \delta_{\omega_0})$,

$$\sum_{n=1}^{\infty} E(|\xi_n|) = \sum_{n=1}^{\infty} |\xi_n(\omega_0)|, \qquad \sum_{n=1}^{\infty} E(\xi_n^2) = \sum_{n=1}^{\infty} \xi_n^2(\omega_0).$$

Now, let $\{\xi_n\}_n$ be a sequence of random variables such that $\sum_{n=1}^{\infty} n^q E(|\xi_n|) < \infty$. So $\sum_{n=1}^{\infty} n^q |\xi_n(\omega_0)| < \infty$. It follows that

$$\sum_{n=1}^{\infty} E(\xi_n^2) = \sum_{n=1}^{\infty} \xi_n^2(\omega_0) \leqq (\sum_{n=1}^{\infty} n^q |\xi_n(\omega_0)|)(\sum_{n=1}^{\infty} n^q |\xi_n(\omega_0)|) < \infty.$$

(2) Let Ω be a non-empty set and $F = \{\emptyset, \Omega\}$. Then the probability space (Ω, F, P) is q-nice for all $q \geq 1$.

The following example shows that there exist probability spaces that are not q-nice for any $q \geq 1$.

Example 2.3. Let $\Omega = [0,1]$ with the σ-field $F = B([0,1])$ of all Borel subsets $B \subseteq [0,1]$ and Lebesgue measure $P = Leb$ on $[0,1]$. Also let

$q \geq 1$. Define $\xi_n : [0,1] \longrightarrow \mathbb{R}$ by

$$\xi_n(\omega) = \begin{cases} n^q & \omega \in [0, \frac{1}{n^{2q+4}}] \\ n^{q+1} & \omega \in (\frac{1}{n^{2q+4}}, \frac{1}{n^{2q+3}}] \\ 0 & \omega \in (\frac{1}{n^{2q+3}}, 1]. \end{cases}$$

One can easily verify that

$$\sum_{n=1}^{\infty} n^q E(|\xi_n|) = \sum_{n=1}^{\infty} n^q E(\xi_n) = \sum_{n=1}^{\infty} (\frac{1}{n^4} + \frac{1}{n^2} - \frac{1}{n^3}) < \infty.$$

But

$$\sum_{n=1}^{\infty} E(\xi_n^2) = \sum_{n=1}^{\infty} (\frac{1}{n^4} + \frac{1}{n} - \frac{1}{n^2}) = \infty.$$

This shows that the probability space $([0,1], B([0,1]), Leb)$ is not q-nice for any $q \geq 1$.

Some basic properties concerning q-nice probability spaces are investigated in [2].

Theorem 2.4. *Let (Ω, F, P) be a q-nice probability space for some $q \geq 1$. Then $L^1(\Omega, F, P) = L^s(\Omega, F, P)$ for all $s \in [1, \infty)$.*

Proof. Clearly $L^2 \subseteq L^1$. Since (Ω, F, P) is q-nice, $L^1 \subseteq L^2$. It follows that $L^2 = L^1$. So $L^1 = L^{2^n}$ for all $n \in \mathbb{N}$. Hence $L^1 = L^s$ for all $s \in [1, \infty)$. □

It is interesting to note that in previous example $\sum_{n=1}^{\infty} E(\xi_n) < \infty$ while $\sum_{n=1}^{\infty} \mathrm{var}(\xi_n) = \infty$. In general, in an arbitrary probability space (Ω, F, P), the condition $E(\xi) < \infty$ does not imply $\mathrm{var}(\xi) < \infty$. But this fact is true in a q-nice probability space.

Proposition 2.5. *Let (Ω, F, P) be a q-nice probability space for some $q \geq 1$, and $\xi : \Omega \longrightarrow \mathbb{R}$ be a random variable such that $E(|\xi|) < \infty$. Then $\mathrm{var}(\xi) < \infty$.*

Proof. As (Ω, F, P) is q-nice, the condition $E(|\xi|) < \infty$ implies that $E(\xi^2) < \infty$. Hence $\mathrm{var}(\xi) = E(\xi^2) - E(\xi)^2 < \infty$. □

Note that in a probability space (Ω, F, P) the equality

$$\sum_{n=1}^{\infty} n^q E(|\xi_n|) = E(\sum_{n=1}^{\infty} n^q |\xi_n|)$$

holds by the monotone convergence theorem [1]. So we can conclude the following results.

Theorem 2.6. *Let (Ω, F, P) be a probability space and $q \in [1, \infty)$. Then (Ω, F, P) is q-nice if and only if for each random variable $\xi : \Omega \longrightarrow \mathbb{R}$, $E(\xi^2) < \infty$ whenever $E(|\xi|) < \infty$.*

Proof. Let $\xi : \Omega \longrightarrow \mathbb{R}$ be a random variable. Then $E(\xi^2) < \infty$ whenever $E(|\xi|) < \infty$. Assume that $\sum_{n=1}^{\infty} n^q E(|\xi_n|) < \infty$, it follows that $E(\sum_{n=1}^{\infty} n^q |\xi_n|) < \infty$. So

$$\sum_{n=1}^{\infty} E(\xi_n^2) \leq E((\sum_{n=1}^{\infty} n^q |\xi_n|)^2) < \infty.$$

□

Corollary 2.7. *Let (Ω, F, P) be a q-nice probability space for some $q \geq 1$. Then it is s-nice for all $s \in [1, \infty)$.*

Corollary 2.8. *Let (Ω, F, P) be a q-nice probability space for some $q \geq 1$. Also let $\{A_n\}_n$ be a sequence of outcomes such that*

$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

Then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(A_n \bigcap A_m) < \infty.$$

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CHARACTER MODULE HOMOMORPHISMS ON NORMED ALGEBRAS

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Abstract. Given normed algebras A and B with $\triangle(B) \neq \emptyset$, the character space of B. We introduce and study a certain class of bounded linear maps from A into B called character module homomorphism.

1. Introduction

Let B be a normed algebra. Then a character on B is a bounded linear map $\varphi : B \longrightarrow \mathbb{C}$ such that $\varphi(bd) = \varphi(b)\varphi(d)$ for all $b, d \in B$. The set of all non-zero characters on B is denoted by $\triangle(B)$ and $\triangle(B) \bigcup \{0\}$ is called the character space of B.

Suppose that V is a non-zero normed vector space and $f \in V^*$(the dual space of V) is a non-zero element such that $\|f\| \leq 1$. For each $a, c \in V$ define $a \cdot c = f(a)c$. The product " $\cdot$ " converts V into an associative algebra that we denote it by V_f. One can easily verify that $\triangle(V_f) = \{f\}$. Some basic properties of V_f such as Arens regularity, amenability, weak amenability, $n-$weak amenability are investigated in

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[1]. Also strongly zero-product, strongly Jordan zero-product, strongly Lie zero-product preserving maps on V_f are investigated in [3, 4, 2, 5].

In this article we introduce a certain class of operators between normed algebras and present some hereditary properties concerning them.

The second dual of a Banach algebra A, denoted A^{**}, is a Banach algebra with respect to the first and the second Arens products $\Box$ and $\Diamond$ respectively. Let us recall the definition: For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$,

$$\langle m\Box n, f\rangle = \langle m, n \cdot f\rangle, \quad \langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle, \quad \langle f \cdot a, b\rangle = \langle f, ab\rangle,$$

and

$$\langle f, m\Diamond n\rangle = \langle f \cdot m, n\rangle, \quad \langle a, f \cdot m\rangle = \langle a \cdot f, m\rangle, \quad \langle b, a \cdot f\rangle = \langle ba, f\rangle.$$

2. Main Results

Definition 2.1. Let A and B be two normed algebras. Then we say that a bounded linear map $T : A \longrightarrow B$ is character module homomorphism if there exists a character $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $g \in B^*$ and $b \in B$.

Example 2.2. Let V be a non-zero normed vector space and $0 \neq f \in V^*$ such that $\|f\| \leq 1$. Also let A be an arbitrary normed algebra. Then every bounded linear map $T : A \longrightarrow V_f$ is a character module homomorphism. Indeed, for $f \in \triangle(V_f) = \{f\}$ we have

$$T^*(g \cdot b) = T^*(f(b)g) = f(b)T^*(g) \qquad (g \in V_f^*, \quad b \in V_f).$$

Proposition 2.3. *Let A and B be two normed algebras. Then the following holds.*

1. *Let $T : A \longrightarrow B$ be a non-zero bounded linear map. If $T^{**} : A^{**} \longrightarrow B^{**}$ is character module homomorphism then so is $T : A \longrightarrow B$.*
2. *Let $T : A \longrightarrow B$ be a non-zero character module homomorphism, then so is $T^{**} : A^{**} \longrightarrow B^{**}$.*

Corollary 2.4. *Let A and B be two normed algebras and let $T : A \longrightarrow B$ be a bounded linear map. Then T is character module homomorphism if and only if for all $n \in \mathbb{N}^* = \{0, 1, 2, \cdots\}$, $T^{(2n)} : A^{(2n)} \longrightarrow B^{(2n)}$ is character module homomorphism, where $A^{(2n)}, B^{(2n)}$ are the $2n^{th}-$dual of A and B respectively.*

Proposition 2.5. *Let A, B, C be normed algebras and let $T : A \longrightarrow B$ be a bounded linear map. Also let $S : B \longrightarrow C$ be a character* *module homomorphism. Then* $S \circ T : A \longrightarrow C$ *is a character module homomorphism.*

Corollary 2.6. *Let* A *be a normed algebra and let* $T : A \longrightarrow A$ *be character module homomorphism. Then for each* $n \in \mathbb{N}^* = \{0, 1, 2, \cdots\}$, $T^n : A \longrightarrow A$ *is character module homomorphism.*

Proposition 2.7. *Let* A *and* B *be two normed algebras and let* $T : A \longrightarrow B$ *be a surjective bounded linear map such that for some* $\varphi \in \triangle(B)$ *the equality* $T^*(g \cdot b) = \varphi(b)T^*(g)$ *, for* $g \in B^*,\ b \in B$*, holds. Then* $B^* \cdot \ker\varphi = \{0\}$*. Moreover,* $\ker\varphi \cdot T(A) = \{0\}$*.*

Let $\mathrm{CMH}(A, B)$ be the set of all non-zero character module homomorphisms from A into B. So for each $T \in \mathrm{CMH}(A, B)$ there exists a unique $\varphi_T \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi_T(b)T^*(g)$ with $g \in B^*,\ b \in B$.

Proposition 2.8. *Let* A *and* B *be two normed algebras such that* $\triangle(B) = \{\varphi\}$*. Then* $\mathrm{CMH}(A, B) \bigcup \{0\}$ *is a closed subspace of* $L(A, B)$ *(the space of all bounded linear maps from* A *into* B*).*

Theorem 2.9. *Let* A *be a normed algebra such that* $\triangle(A) = \{\psi\}$*. Then* $\mathrm{CMH}(A) \bigcup \{0\} = \mathrm{CMH}(A, A) \bigcup \{0\}$ *is a closed right ideal of* $L(A) = L(A, A)$*.*

For $T, S \in \mathrm{CMH}(A, B)$ define $T \sim S$ if and only if $\varphi_T = \varphi_S$.

Proposition 2.10. *Let* A *and* B *be two normed algebras such that* $\triangle(B) \neq \emptyset$*. Then* $\sim$ *is an equivalence relation on* $\mathrm{CMH}(A, B)$*.*

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CONTRACTIBILITY AND WEAK AMENABILITY OF HYPERGROUP ALGEBRAS AND THEIR SECOND DUAL

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Abstract. Let K be a hypergroup. We prove that finiteness of K is equivalent to the contractibility of $L(K)$. Also, we study approximate amenability and weak amenability of the hypergroup algebras $(L(K)^* \cdot L(K))^*$ and $L(K)^{**}$.

1. Introduction

For a locally compact Hausdorff space K, let $M(K)$ be the Banach space of all bounded complex regular Borel measures on K. For $x \in K$, the unit point mass at x will be denoted by δ_x. Let $M_1(K)$ be the set of all probability measures on K, $C_b(K)$ be the Banach space of all continuous bounded complex-valued functions on K. We denote by $C_0(K)$ the space of all continuous functions on K vanishing at infinity and by $C_c(K)$ the space of all continuous functions on K with compact support.

The space K is called a hypergroup if there is a map $\lambda : K \times K \longrightarrow M_1(K)$ with the following properties:

(i) For every $x, y \in K$, the measure $\lambda_{(x,y)}$ (the value of λ at (x, y)) has a compact support.

(ii) For each $\psi \in C_c(K)$, the map $(x, y) \longmapsto \psi(x * y) = \int_K \psi(t) d\lambda_{(x,y)}(t)$ is in $C_b(K \times K)$ and $x \longmapsto \psi(x * y)$ is in $C_c(K)$, for every $y \in K$.

(iii) The convolution $(\mu, \nu) \longmapsto \mu * \nu$ of measures defined by

$$\int_K \psi(t) d(\mu * \nu)(t) = \int_K \int_K \psi(x * y) d\mu(x) d\nu(y).$$

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* Speaker.

is associative where $\mu, \nu \in M(K), \psi \in C_0(K)$ (note that $\lambda_{(x,y)} = \delta_x * \delta_y$).

(iv) There is a unique point $e \in K$ such that $\lambda_{(x,e)} = \delta_x$, for all $x \in K$.

When $\lambda_{(x,y)} = \lambda_{(y,x)}$, we say that K is a commutative hypergroup, for more details see [4, 6, 10].

Let K be foundation, i.e., $K = \mathrm{cl}(\bigcup_{\mu \in L(K)} \operatorname{supp} \mu)$, we define

$$L(K) = \{\mu \mid \mu \in M(K), x \mapsto |\mu| * \delta_x, x \mapsto \delta_x * |\mu| \text{ are norm-continuous}\}.$$

It is easy to see that $L(K)$ is an ideal in $M(K)$. Also, if K admits an invariant measure (Haar measure m) then $L(K) = L^1(K, m)$ [6].

An involution on a hypergroup K is a homeomorphism $x \mapsto \tilde{x}$ in K such that $\tilde{\tilde{x}} = x$ and $e \in \operatorname{supp} \lambda_{(x,\tilde{x})}$ for all $x \in K$. For each $\mu \in M(K)$, define $\widetilde{\mu} \in M(K)$ by $\widetilde{\mu}(A) = \overline{\mu(\tilde{A})}$, i.e., $\int_K f(x) d\widetilde{\mu}(x) = \int_K f(\tilde{x}) d\overline{\mu}(x)$, for each $f \in C_c(K)$. Then $\mu \longrightarrow \tilde{\mu}$ is an involution on $M(K)$ such that $M(K)$ and $L(K)$ are Banach *-algebras [5] and $\widetilde{\lambda}_{(x,y)} = \lambda_{(\tilde{y},\tilde{x})}$, whenever $x, y \in K$ [4].

Let K is a foundation hypergroup without a Haar measure. With these conditions $L(K)$ is very general hypergroup algebras which include not only group algebras but also most of semigroup algebras. In [10], it has been shown that $(L(K)^* \cdot L(K))^*$ (dual of $L(K)^* \cdot L(K)$) is a Banach algebra by an Arens type product and that $L(K) \subseteq (L(K)^* \cdot L(K))^*$. For $f \in L(K)^* \cdot L(K)$, if K admits an invariant measure (Haar measure m), then by Proposition 2.4 of [11], $L(K)^* \cdot L(K) = LUC(K)$ where

$$LUC(K) = \{f \mid f \in C_b(K), x \to l_x f \text{ from } K \text{ into } C_b(K) \text{ is continuous}\},$$

and $l_x f(y) = f(x * y)$, for any $y \in K$.

2. Main results

Throughout this paper, K is a foundation hypergroup without a Haar measure.

Now, we are in a position to prove a theorem generalizes one side of Theorem 3.2 of [7] to hypergroups.

Theorem 2.1. *Let K be a hypergroup with an involution. Then $L(K)$ is contractible if and only if K is finite.*

Proof. Let $L(K)$ be contractible. By Theorem 2.8.48 of [3], $L(K)$ is biprojective and unital. Therefore, K is discrete. Since K is discrete and has an involution, Jewett's definition and Dunkl's definition of hypergroup coincide. It follows that K has a Haar measure and $\ell^1(K) = L(K)$ [5, Theorem 7.1.A]. Now, since $L(K)$ is biprojective and $\mathbb{C}$ is an essential module over $L(K)$, $\mathbb{C}$ is projective [9, Proposition 5.3]. On the other hand, the map $\varphi_K : L(K) \longrightarrow \mathbb{C}$ defined by $\varphi_K(\mu) = \mu(K)$ is admissible. Therefore, φ_K has a right inverse morphism ρ. Take $P_0 := \rho(1) \in L(K)$, so

$$f * P_0 = f * \rho(1) = \rho(f.1) = \rho(\varphi_K(f)),$$

for any $f \in L(K)$. Now, suppose that $f \in C_c^+(K)$ and $\|f\|_1 = 1$. Then, $\|l_x f\|_1 = 1$, where $l_x f(y) = f(x * y)$ for all $x, y \in K$. We have

$$\begin{aligned} \|l_x f * P_0 - l_x P_0\|_1 &= \|l_x(f * P_0) - l_x P_0\|_1 = \|l_x(f * P_0 - P_0)\|_1 \\ &\leq \|f * P_0 - P_0\|_1 = 0. \end{aligned}$$

Hence, $P_0 = l_x f * P_0 = l_x P_0$ almost everywhere. Since $\varphi_K(P_0) = \varphi_K(\rho(1)) = 1$, P_0 is equal to a nonzero constant almost everywhere. It follows that the characteristic function $1_K \in L(K)$, since $P_0 \in L(K)$. On the other hand, $c1_K = 1_K * 1_K \in C_0(K)$ where $c > 0$ [2, Proposition 1.4.11]. Thus, K is compact. This follows that K is finite.

Conversely, K is finite. So, $\ell^1(K)$ is amenable [1, Theorem 3.3]. Therefore, there exists $M \in (\ell^1(K)\hat{\otimes}\ell^1(K))^{**}$ such that M is a virtual diagonal for $\ell^1(K)$. On the other hand, $(\ell^1(K)\hat{\otimes}\ell^1(K))^{**} = \ell^1(K)\hat{\otimes}\ell^1(K)$. It follows that M is a diagonal for $\ell^1(K)$. Thus, $\ell^1(K)$ is contractible [3, Theorem 1.9.21]. $\square$

In this paper, the second dual $L(K)^{**}$ with the first Arens product denoted by $(L(K)^{**}, \square)$. Also, $\pi : L(K)^{**} \longrightarrow (L(K)^* \cdot L(K))^*$ is the adjoint of embedding of $L(K)^* \cdot L(K)$ in $L(K)^*$.

By a well-known result of Ghahramani et al. [8], if $L^1(G)^{**}$ is weakly amenable, then $M(G)$ is weakly amenable. The following theorem extends this result to hypergroups.

Theorem 2.2. *Let K be a hypergroup. Then*

(i) *If $(L(K)^* \cdot L(K))^*$ is weakly amenable, then $M(K)$ is weakly amenable.*
(ii) *If $(L(K)^{**}, \square)$ is approximately amenable (weakly amenable), then $M(K)$ is approximately amenable (weakly amenable).*

Proof. (i). For $f \in M(K)^*$, define $T_f \in (L(K)^* \cdot L(K))^{**}$ by $\langle T_f, \mu + m\rangle = f(\mu)$, where $\mu \in M(K)$ and $m \in C_0(K)^\perp$ $((L(K)^* \cdot L(K))^* = M(K) \oplus C_0(K)^\perp)$. Assume that $M(K)$ is not weakly amenable. So, there is a non-inner derivation $D : M(K) \longrightarrow M(K)^*$. Define $\triangle : (L(K)^* \cdot L(K))^* \longrightarrow (L(K)^* \cdot L(K))^{**}$ by $\triangle(\mu + m) = T_{D(\mu)}$, for each $\mu \in M(K), m \in C_0(K)^\perp$. For each $\mu_1, \mu_2, \nu \in M(K)$ and $m_1, m_1, n \in C_0(K)^\perp$, we have

$$\begin{aligned}
\langle(\mu_1 + m_1) \triangle (\mu_2 + m_2), \nu + n\rangle &= \langle\triangle(\mu_2 + m_2), (\nu + n)(\mu_1 + m_1)\rangle \\
&= \langle\triangle(\mu_2 + m_2), \nu * \mu_1 + n\mu_1 + \nu m_1 + n\square m_1\rangle \\
&= \langle T_{D(\mu_2)}, \nu * \mu_1 + n\mu_1 + \nu m_1 + n\square m_1\rangle \\
&= \langle D(\mu_2), \nu * \mu_1\rangle = \langle\mu_1 D(\mu_2), \nu\rangle \\
&= \langle T_{(\mu_1 D(\mu_2))}, \nu + n\rangle,
\end{aligned}$$

since $C_0(K)^\perp$ is a closed ideal of $(L(K)^* \cdot L(K))^*$. It follows that $(\mu_1 + m_1) \triangle (\mu_2 + m_2) = T_{(\mu_1 D(\mu_2))}$. By a similar argument, $\triangle(\mu_2 + m_2)(\mu_1 + m_1) = T_{(D(\mu_2)\mu_1)}$. Therefore,

$$\begin{aligned}
\triangle[(\mu_2 + m_2)(\mu_1 + m_1)] &= T_{D(\mu_2 * \mu_1)} = T_{[D(\mu_2)\mu_1 + \mu_2 D(\mu_1)]} \\
&= \triangle(\mu_2 + m_2)(\mu_1 + m_1) + (\mu_1 + m_1) \triangle (\mu_2 + m_2).
\end{aligned}$$

It follows that Δ is a derivation and $\Delta|_{M(K)} = D$. We prove that Δ can not be inner. If Δ is inner, then there is an $F \in (L(K)^* \cdot L(K))^{**}$ such that $\Delta(G) = GF - FG$, for all $G \in (L(K)^* \cdot L(K))^*$. If $\Psi := G|_{M(K)}$, then Ψ is an element of $M(K)^*$. Now, for all $\mu \in M(K)$, we have

$$D(\mu) = \Delta(\mu) = \mu\Psi - \Psi\mu.$$

Hence, in contradiction with D is not inner. It follows that $(L(K)^* \cdot L(K))^*$ is not weakly amenable.

(ii). $L(K)$ has a bounded approximate identity $(e_\alpha)_\alpha$ with $\|e_\alpha\| = 1$ [6, Lemma 1]. Let E be a *weak** cluster point of (e_α) in $L(K)^{**}$, it is clear that E is a right identity for $L(K)^{**}$ and $\| E \|= 1$ [10, Lemma 5]. The map

$$\varphi : L(K)^{**} \longrightarrow E\Box L(K)^{**}, \qquad F \longmapsto E\Box F,$$

is an epimorphism. On the other hand $L(K)^{**}$ is approximately amenable, therefore $E\Box L(K)^{**}$ is approximately amenable [7, Proposition 2.2]. By Theorems 7 and 4 of [10], $E\Box L(K)^{**}$ is isometrically isomorphic to $(L(K)^* \cdot L(K))^* = M(K) \oplus C_0(K)^{\perp}$, where $C_0(K)^{\perp}$ is a closed ideal in $(L(K)^* \cdot L(K))^*$ and $C_0(K)^{\perp} = \{m \in (L(K)^* \cdot L(K))^* \mid \text{for all } f \in C_0(K), \langle m, f\rangle = 0\}$. Thus, $M(K)$ is approximately amenable [7, Corollary 2.1].

Now, let $(L(K)^{**}, \Box)$ be weakly amenable and $M(K)$ is not weakly amenable. Then, by a similar argument of the part of (i), the derivation $\Delta : (L(K)^* \cdot L(K))^* \rightarrow (L(K)^* \cdot L(K))^{**}$ is not inner. Now, let E be a right identity of $L(K)^{**}$. $E\Box L(K)^{**}$ is isometrically isomorphic to $(L(K)^* \cdot L(K))^*$, therefore we may consider Δ to be defined on $E\Box L(K)^{**}$. Now, define $\Lambda : L(K)^{**} \longrightarrow L(K)^{***}$ by $\Lambda(G) = \Delta(E\Box G)$, for all $G \in L(K)^{**}$. Since $L(K)^{**} = E\Box L(K)^{**} + (1-E)\Box L(K)^{**}$, Λ is a non-inner derivation (see [8]). It follows that $L(K)^{**}$ is not weakly amenable and this is a contradiction of the hypothesis. Therefore, $M(K)$ is weakly amenable. $\Box$

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INNERNESS OF σ-DERIVATIONS, INNERNESS OF σ-HIGHER DERIVATIONS AND W^*-ALGEBRA

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Abstract. Let $\mathcal{M}$ be a W^*-algebra and $\mathcal{A} \subseteq \mathcal{M}$ a commutative W^*-subalgebra of $\mathcal{M}$ containing the identity element of $\mathcal{M}$. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a continuous homomorphism and $\delta : \mathcal{M} \to \mathcal{M}$ a σ-derivation. In this paper, it is proved that there exists a $x_0 \in \mathcal{M}$ such that $\delta(a) = \sigma(a)x_0 - x_0\sigma(a)$, for each $a \in \mathcal{A}$. Also, it is shown that if $\{d_n\}$ is a continuous strongly σ-higher derivationon on $\mathcal{M}$, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$\begin{aligned} d_n(a) =&\sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) \\ &- x_{n-2}d_2(\sigma^{n-2})(a) - \ldots - x_1 d_{n-1}(\sigma(a)) \end{aligned}$$

for each $a \in \mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be an algebra. We say that a linear mapping d on $\mathcal{A}$ is a derivation if it satisfies

$$d(ab) = ad(b) + d(a)b \qquad (a, b \in \mathcal{A}).$$

It also is said to be inner, if there exists $x_0 \in \mathcal{A}$ such that $d(a) = ax_0 - x_0 a$ for all $a \in \mathcal{A}$.

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A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b)$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer n; we say that it is strongly, if $d_0 = I$.

In view of [5, Theorem 2.5.1], if $\mathcal{M}$ is a W^*-algebra and $\mathcal{A} \subseteq \mathcal{M}$ is a commutative W^*-subalgebra containing the identity element of $\mathcal{M}$ and $d : \mathcal{M} \to \mathcal{M}$ a derivation, then there exists a $x_0 \in \mathcal{M}$ such that $d(a) = ax_0 - x_0a$ for each $a \in \mathcal{A}$. Also in light of [5, Theorem 2.5.3], every derivation on a W^*-algebra is inner. In view of [4], if $\mathcal{M}, \mathcal{A}$ are as in the same sets stated above and $\{d_n\}$ a stongly higher derivation, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$\begin{aligned} d_n(a) =& ax_n - x_na - x_{n-1}d_1(a) \\ & - x_{n-2}d_2(a) - ... - x_1d_{n-1}(a), \qquad (a \in \mathcal{A}). \end{aligned}$$

Now, in what follows, assume that σ, τ are two homomorphism on $\mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a (σ, τ)-derivation if it satisfies the generalized Leibniz rule $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for each $x, y \in \mathcal{A}$. By a σ-derivation, we mean a (σ, σ)-derivation. An ordinary derivation is an I-derivation where I is the identity mapping on $\mathcal{A}$.

2. Results

Theorem 2.1. *Let $\mathcal{M}$ be a W^*-algebra and $\mathcal{A} \subseteq \mathcal{M}$ a commutative W^*-subalgebra of $\mathcal{M}$ containing the identity element of $\mathcal{M}$. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a continuous homomorphism and $d : \mathcal{M} \to \mathcal{M}$ a σ-derivation. There exists an element $x_0 \in \mathcal{M}$ with the property $\|x_0\| \leq \|\sigma\|\|d\|$ such that $d(a) = \sigma(a)x_0 - x_0\sigma(a)$ for each $a \in \mathcal{A}$.*

Proof. In view of [3], d is continuous. From [1, Proposition 14], $\mathcal{A}$ is the linear span of its unitary elements (i.e., any element of $\mathcal{A}$ is a finite linear combination of the unitary elements). Suppose $\mathcal{A}^u$ be the set of all unitary elements of $\mathcal{A}$. It is enough to show that the result holds for $\mathcal{A}^u$. For each $u \in \mathcal{A}^u$, we define the linear mapping

$$T_u : \mathcal{M} \to \mathcal{M}, \quad T_u(x) = (\sigma(u))^{-1}(x\sigma(u) + d(u)), \quad (x \in \mathcal{M}).$$

Suppose $a, b \in \mathcal{A}^u$. One can show that for each x

$$T_aT_b(x) = T_{ba}(x) = T_{ab}(x) = T_aT_b(x).$$

Now consider $\Omega = \{T_u : u \in \mathcal{A}^u\}$; for each $u \in \mathcal{A}^u$, the function T_u is σ-continuous. Suppose $\mathcal{K}$ is the σ-closed convex subset of $\mathcal{M}$ generated by $\{T_u(0) : u \in \mathcal{A}^u\}$. Since

$$\|T_u(0)\| \leqslant \|\sigma\|\|d\|,$$

in light of the Banach-Alaoglu theorem, $\mathcal{K}$ is σ-compact. Also since $T_{vu} = T_v T_u$ for each $u, v \in \mathcal{A}^u$, it is immediate that $T(\mathcal{K}) \subseteq \mathcal{K}$. Thus, in view of the Markov-Kakutani theorem (see [2, Theorem VII.2.1]), there exists $x_0 \in \mathcal{K}$ such that $T(x_0) = x_0$ for each $T \in \Omega$; actually, for each $u \in \mathcal{A}^u$, $T_u(x_0) = x_0$. In fact, for each $u \in \mathcal{A}^u$,

$$d(u) = \sigma(u)x_0 - x\sigma(u).$$

Moreover, it is obvious

$$\|x_0\| \leq \|\sigma\|\|d\|.$$

□

Remark 2.2. Let $\sigma, \tau : \mathcal{M} \to \mathcal{M}$ be continuous homomorphisms and $\delta : \mathcal{M} \to \mathcal{M}$ a (σ, τ)-derivation. Then, similar to the proof of Theorem 2.1, and by considering

$$T_u : \mathcal{M} \to \mathcal{M}, \quad T_u(x) = (\sigma(u))^{-1}(x\tau(u) + d(u)), \quad (x \in \mathcal{M})$$

one can show that there exists an element $x_0 \in \mathcal{M}$ such that $d(a) = \tau(a)x_0 - x_0\sigma(a)$ for each $a \in \mathcal{A}$.

Definition 2.3. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a homomorphism; in view of [3], a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a σ-higher derivation if it satisfies

$$d_n(ab) = \sum_{k=0}^{n} d_k(\sigma^{n-k}(a))d_{n-k}(\sigma^k(b))$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer n; it is strongly if $d_0 = I$. We say that a stongly σ-higher derivation $\{d_n\}$ is inner if there exist $x_1, x_2, x_3, ...x_n \in \mathcal{A}$ such that

$$\begin{aligned} d_n(a) =& \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) \\ & - x_{n-2}d_2(\sigma^{n-2}(a) - \ldots - x_1 d_{n-1}(\sigma(a)). \end{aligned}$$

If, moreover, $\tau : \mathcal{M} \to \mathcal{M}$ is a homomorphism, a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a (σ, τ)-higher derivation if it satisfies

$$d_n(ab) = \sum_{k=0}^{n} d_k(\tau^{n-k}(a))d_{n-k}(\sigma^k(b))$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer n; we say that a stongly (σ, τ)-higher derivation $\{d_n\}$ is inner if there exist $x_1, x_2, x_3, \ldots, x_n \in \mathcal{A}$ such that

$$\begin{aligned} d_n(a) =& \tau^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) \\ & - x_{n-2}d_2(\sigma^{n-2}(a) - \ldots - x_1 d_{n-1}(\sigma(a)) \end{aligned}$$

for each $a \in \mathcal{A}$.

Theorem 2.4. *Let $\mathcal{M}, \mathcal{A}$ be as in Theorem 2.1. Let $\{d_n\}$ be a continuous and strongly σ-higher derivation on $\mathcal{A}$. Then there exist elements $x_1, x_2, x_3, \ldots, x_n \in \mathcal{M}$ with the property*

$$\|x_n\| \leqslant \|d_n\|\|\sigma\| + \Big(\|x_1\|\|d_{n-1}\| + \ldots + \|x_{n-1}\|\|d_1\|\Big)\|\sigma\|^2$$

such that

$$\begin{aligned} d_n(a) =& \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) \\ & - x_{n-2}d_2(\sigma^{n-2}(a) - \ldots - x_1 d_{n-1}(\sigma(a)) \end{aligned}$$

for each $a \in \mathcal{A}$.

Proof. We proceed by induction and follow the same line as to the proof of Theorem 2.1. □

Remark 2.5. The analogue of Theorem 2.4 can be stated for (σ, τ)-higher derivations.

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A GENERALIZATION OF A QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we first investigate solutions for the functional equations

$$f(2x+y)+f(2x-y)=g(x+y)+g(x-y)+h(x)+e(y) \qquad (*)$$

and

$$f(2x+y)-f(2x-y)=g(x+y)-g(x-y)+o(y), \qquad (**)$$

in which $f,g,h,e,o:\mathbb{R}\to\mathbb{R}$ are functions with $e(0)=0$. Then, using our results, we give the solutions of the functional equation (***)

$$f_1(2x+y)+f_2(2x-y)=f_3(x+y)+f_4(x-y)+f_5(x)+f_6(y).$$

As a special case, we deal with the quartic functional equation (****)

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

where $f_i, f:\mathbb{R}\to\mathbb{R}$.

1. Introduction

A function $A:\mathbb{R}\to\mathbb{R}$ is said to be additive if $A(x+y)=A(x)+A(y)$ for all $x,y\in\mathbb{R}$. A function $A_n:\mathbb{R}^n\to\mathbb{R}$, $n\in\mathbb{N}$, that is additive in each

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of its variables, is called n-additive. Substituing $x_1 = x_2 = ... = x_k = x$ and $x_{k+1} = x_{k+2} = ... = x_n = y$ in $A_n(x_1, x_2, ..., x_n)$, the resulting function will be denoted by $A^{k,n-k}(x, y)$. We say that a function $A_n : \mathbb{R}^n \to \mathbb{R}$ is symmetric if $A_n(x_1, x_2, ..., x_n) = A_n(x_{\alpha(1)}, x_{\alpha(2)}, ..., x_{\alpha(n)})$ for every permutation$\{\alpha(1), \alpha(2), ...\alpha(n)\}$ of $\{1, 2, ..., n\}$. If $A_n(x_1, x_2, ..., x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, ..., x)$; for more about these objects, see [1, 2].

The difference operator Δ_h with the span $h \in \mathbb{R}$ is defined by

$$\Delta_h f(x) = f(x+h) - f(x), \qquad (x \in \mathbb{R}, \quad f : \mathbb{R} \to \mathbb{R}). \tag{1.1}$$

In fact, Δ_h is an operator. Furthermore, for every nonnegative integer n, the notation Δ_h^n is defined as

$$\Delta_h^0 f(x) = f(x), \quad \Delta_h^1 f(x) = \Delta_h f(x), \quad \Delta_h^n f(x) = \Delta_h \circ \Delta_h^{n-1} f(x)$$

in which the notation $\circ$ is the composition operation of functions. Moreover, for each $h_1, h_2 \in \mathbb{R}$, the notation Δ_{h_1,h_2} is considered as $\Delta_{h_2} \circ \Delta_{h_1}$; for more about these facts, see [1, 2].

For any given $n \in \mathbb{N}$, if f satisfies the functional equation

$$\Delta_h^{n+1} f(x) = 0 \qquad (x, h \in \mathbb{R}), \tag{1.2}$$

then f is called a polynomial function of order n. It is equivalent to the fact that

$$\Delta_{h_1,h_2,...,h_{n+1}} f(x) = 0, \qquad (x, h_1, h_2, ..., h_{n+1} \in \mathbb{R}); \tag{1.3}$$

in fact, f satisfies (1.2) if and only if it satisfies (1.3); see [1, Theorem 9.3].

Theorem 1.1 (Theorem 9.3. and 9.6 in [1])**.** *$f : \mathbb{R} \to \mathbb{R}$ satisfies (1.3) if and only if for every $x \in \mathbb{R}$, $f(x) = \sum_{i=0}^{n} A^i(x)$ where $A^0(x) = A^0$ is an arbitrary constant and A^n is the diagonal n-additive symmetric map $A_n : \mathbb{R}^n \to \mathbb{R}$.*

2. RESULTS

Theorem 2.1. *Let f be a function satisfying (*). Then $f = \sum_{i=0}^{4} A^i$, where $A^0(x) = A^0$ is an arbitrary constant and A^n is the diagonal n-additive symmetric map $A_n : \mathbb{R}^n \to \mathbb{R}$.*

Proof. One can show that

$$\Delta_{x_1,x_2,...,x_5} f(x) = 0 \qquad (x, x_1, x_2, ..., x_5 \in \mathbb{R}).$$

Hence, from Theorem 1.1, $f = \sum_{i=0}^{4} A^i$. $\square$

Theorem 2.2. *Let f be a function satisfying (**). Then $f = \sum_{i=0}^{3} A^i$.*

Proof. It is similar to the proof of Theorem 2.1. $\square$

Corollary 2.3. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies (***) if and only if it is quartic.*

Proof. From Theorem 2.1, $f = \sum_{i=0}^{4} A^i$ satisfies (****). By substituting it in (****), one can conclude that $A^0 = A^1 = A^2 = A^3 = 0$. Hence $f = A^4$. The converse is routine. □

Theorem 2.4. *The functions f, g, h, e satisfy (*) with $e(0) = 0$ if and only if*

$$f = \textstyle\sum_{i=0}^{4} A^i, \quad g = \sum_{i=0}^{2} B^i + 4A^4 + 2A^3 + A^2 + A^1 + A^0,$$
$$h = -2\textstyle\sum_{i=0}^{2} B^i + 24A^4 + 12A^3 + 6A^2 + 2A^1, \quad e = -2B^2 - 6A^4.$$

Proof. In view of Theorem 2.1, $f = \sum_{i=0}^{4} A^i$ satisfies (*). Then, substituing f in (*) and defining

$$G = g - 4A^4 - 2A^3 - A^2 - A^1 - A^0, \tag{2.1}$$

$$H = -h + 24A^4 + 12A^3 + 6A^2 + 2A^1 \tag{2.2}$$

and

$$E = -e - 6A^4, \tag{2.3}$$

we arrive at

$$G(x + y) + G(x - y) = H(x) + E(y). \tag{2.4}$$

One can prove that $G = \sum_{i=0}^{2} B^i$, $H(x) = 2G(x)$ and $E = 2B^2$. Then from (2.1), (2.2) and (2.3), we get the result. The converse direction is routine. □

Theorem 2.5. *The functions f, g, o satisfy (**) if and only if*

$$f = \sum_{i=0}^{3} A^i, \quad g = \sum_{i=0}^{1} B^i + 4A^3 + 2A^2 + A^1, \quad o = -2B^1 - 6A^3.$$

Proof. It is similar to the proof of Theorem 2.4. □

Theorem 2.6. *The functions $f_1, f_2, f_3, f_4, f_5, f_6$ satisfy (***) if and only if*

$$f_1 = \frac{1}{2}\sum_{i=0}^{4} A^i + \frac{1}{2}\sum_{i=0}^{3} C^i, \quad f_2 = \frac{1}{2}\sum_{i=0}^{4} A^i - \frac{1}{2}\sum_{i=0}^{3} C^i,$$

$$f_3 = \frac{1}{2}(\sum_{i=0}^{2} B^i + 4A^4 + 2A^3 + A^2 + A^1 + A^0)$$

$$+\frac{1}{2}(\sum_{i=0}^{1} D^i + 4C^3 + 2C^2 + C^1),$$

$$f_4 = \frac{1}{2}(\sum_{i=0}^{2} B^i + 4A^4 + 2A^3 + A^2 + A^1 + A^0)$$

$$-\frac{1}{2}(\sum_{i=0}^{1} D^i + 4C^3 + 2C^2 + C^1),$$

$$f_5 = -\sum_{i=0}^{2} B^i + 12A^4 + 6A^3 + 3A^2 + A^1,$$

$$f_6 = -B^2 - 3A^4 - D^1 - 3C^3.$$

Proof. With the aid of Theorems 2.2, and 2.4, one can conclude the asserted solutions. The converse is routine. $\square$

Remark 2.7.
(1) One can talk regarding the solutions of the functional equation (***) on the commutative groups with some additional requirements and also the linear spaces over the field $\mathbb{Q}$; see [1, 3].
(2) It might be well to point out that, in connection with the concept of the stability of functional equations, we need to know the solution or solutions of the equations; the method stated in this paper is a very well technique for this purpose (of course, for polynomial functional equations).

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ON THE CHARACTERIZATION OF MAJORIZATION AND ITS PROPERTIES IN INFINITE DIMENSIONAL

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Abstract. In this note, we consider two definitions for majorization. One of the definitions is in the term of rearrangement and the other definition is based on doubly stochastic operators. With respect to the entanglement transformation problem in quantum information theory, linking majorization and trumping order is useful.

1. Introduction

Majorization was scattered in journals in a wide variety of fields, specially in quantum mechanics. Although much of the work on the quantum information theory is limited to the finite dimensional cases, but in quantum mechanics we need to work in infinite dimension.

Let x, y be tow vectors in $\mathbb{R}^n$, Hardy, Littlewood and Polya proved that $x \prec y$ if and only if there is a doubly stochastic matrix D such that $x = Dy$. Recently, Bahrami, Bayati and Manjegani extended this result to the concept of majorization on the Banach space $l^p(I)$, where I is an arbitrary non-empty set and $p \in [1, \infty)$ [1, 2, 3].

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Definition 1.1 ([1]). For any $x, y \in l^1$, we say that x is majorized by y and denote by $x \prec y$ when there is a doubly stochastic operator D such that $x = Dy$.

Also, Owari, Braunstein, Nemoto and Murao extended notion of majorization in [4] which the best fits the physical descriptions of infinite dimensional quantum states.

So, based on the majorization in the physical applications in infinite dimensional, we will work on l^1_+ that includes all $x = \{x_n\}_{n=1}^{\infty} \in l^1$ such that $x_n \geqslant 0$ for all $n \in \mathbb{N}$ and exactly one of the sets $\{n \in \mathbb{N} : x_n > 0\}$ and $\{n \in \mathbb{N} : x_n = 0\}$ is finite.

Definition 1.2 ([5]). Let $x, y \in l^1_+$. We say that x is majorized by y and write $x \prec y$ if and only if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} y_i^{\downarrow} \qquad (k \in \mathbb{N})$$

and

$$\sum_{i=1}^{\infty} x_i^{\downarrow} = \sum_{i=1}^{\infty} y_i^{\downarrow},$$

where $x^{\downarrow}$ symbolizes that the components of x are rearranged in non-increasing order: $x_i^{\downarrow} \geq x_{i+1}^{\downarrow}$ for all $i \geq 1$.

2. Main results

First, we recall that $\max(z, 0)$ is refer to the symbol z^+ whenever $z \in \mathbb{R}$. In the following proposition the notation $\prec$ is in the sense of Definition 1.2.

Proposition 2.1 ([4]). *Let $a = \{a_n\}_{n=1}^{\infty}$ and $b = \{b_n\}_{n=1}^{\infty}$ both be in l^1_+. Then $a \prec b$ if and only if the following two conditions are satisfied:*

(1) $\sum_{i=1}^{\infty} (a_i - t)^+ \leqslant \sum_{i=1}^{\infty} (b_i - t)^+$ *for all* $t > 0$.
(2) $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i$.

Before we state our main theorem, we need to recall the concept of completely monotone functions.

Definition 2.2 ([5]). Let $I \subset \mathbb{R}$. A function f is said to be completely monotone on I if $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x \in I$ and all $n = 0, 1, 2, \ldots$.

Theorem 2.3 ([1]). *Let $f, g \in l^p$ and $f \prec g$, in the sense of Definition 1.1, then for any nonnegative convex function ϕ with*

$$\phi : (a, b) \subset \mathbb{R} \to \mathbb{R}$$

we have

$$\sum_{i\in I}\phi(f_i)\leqslant\sum_{i\in I}\phi(g_i).$$

Using Bernstein's theorem, we see that the necessary and sufficient condition for the function f to be completely monotone on $(0,\infty)$ is that f be the Laplace transform of a positive measure μ,i.e.,

$$f(s)=\int_0^\infty e^{-st}d\mu(t).$$

Recall that the Mellin transform of a function f on $(0,\infty)$ is the function $\phi(s)=\int_0^\infty f(t)t^{s-1}dt$. There is a close relationship between the Mellin and Laplace transforms, in fact if $f\in L^1(0,\infty)$ is zero outside of $[0,1]$, then the Mellin transform of $f(x)$ is the Laplace transform of $f(e^{-x})$. Thus, if $f\in L^1(0,\infty)$ is zero outside of $[0,1]$, then the necessary and sufficient condition for the Mellin transform of f to be completely monotone on $(0,\infty)$ is that f be an almost everywhere nonnegative function. Pereira and Plosker used this fact inthe proof of Theorem 2 in [5]. We will use this fact here.

Now we are ready to introduce our results.

Theorem 2.4. *Let $a=\{a_n\}_{n=1}^\infty$ and $b=\{b_n\}_{n=1}^\infty$ both be in l_+^1 and set $\zeta(s)=\sum_{n=1}^\infty b_n^s-\sum_{n=1}^\infty a_n^s$. If there is a doubly stochastic operator D such that $a=Db$, then the following two conditions are satisfied:*

(1) $\zeta(1)=0$.

(2) $\dfrac{\zeta(s)}{s(s-1)}$ *is completely monotone on* $(1,\infty)$.

Using Theorem 2.4 and Theorem 2 in [5], we can obtain the following corollary.

Corollary 2.5. *Let $x,y\in l_+^1$. If $x\prec y$ in the sense of Definition 1.1 then $x\prec y$ in the sense of Definition 1.2.*

If we can prove that two Definitions 1.1 and 1.2 are equivalent (even under some extra conditions), then doubly stochastic operators will play an important role in the solving entangelment transformation problem, that is, one of the most important problems in quantum information theory. To better understand, we need the following definition.

Definition 2.6. For any $x,y\in l_+^1$, we say that x is trumped by y with the symbol $x\prec_T y$, if there exists a vector $c\in l_+^1$ such that $||c||=1$ and $x\otimes c\prec y\otimes c$. We say that c is a catalyst.

In fact, with respect to the entanglement transformation problem in quantum information theory, for two arbitrary vectors that whether or not there is a majorization relationship between them, we should determine the existence conditions and identification for catalyst, as showed in the following conjecture[5] (see Theorem 2.4):

Conjecture ([5]): The function $\zeta(s)$ is positive on $(1, \infty)$ if and only if there exists a catalyst $c = \{c_n\}_{n=1}^{\infty} \in l_+^1$ with corresponding generalized Dirichlet series $\zeta_2(s) = \sum_{n=1}^{\infty} c_n^s$ such that $x \otimes c \prec y \otimes c$.

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POWER REGULARITY OF d-TUPLE OF OPERATORS

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Abstract. A bounded linear operator S on a Hilbert space $\mathcal{H}$ is power regular if $\lim_{n\to\infty} \|S^n x\|^{1/n}$ exists for every $x \in \mathcal{H}$. In this talk, we define the concept of power regularity for commuting tuples of opertors and prove that the spherical m-isometries are power regular. Moreover, we provide conditions on a right invertible spherical isometry making it a spherical unitary.

1. Introduction

A bounded linear operator S in Hilbert space $\mathcal{H}$ is power regular if $\lim_{n\to\infty} \|S^n x\|^{1/n}$ exists for every $x \in \mathcal{H}$. It is known that compact operators, hyponormal operators, decomposable operators and isometric N-Jordan operators are power regular. The backward shift operator is an example of an operator that is not power regular. Power regularity helps us to find a non-trivial invariant subspace of an operator. For example, if S is hyponormal and $\lim_{n\to\infty} \|S^n x\|^{1/n} < \|S\|$ for some non-zero vector x then x is not a cyclic vector for S; i.e., $\bigvee_{n\geq 0}\{S^n x\}$

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* Speaker.

is a non-trivial invariant subspace of S [2]. For some information on power regularity of operators one can see [1, 5] and references there in.

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we set $|\alpha| = \sum_{j=1}^{d} \alpha_j$, $\alpha! = \alpha_1! \ldots \alpha_d!$ and further $T^\alpha = T_1^{\alpha_1} \ldots T_d^{\alpha_d}$. For every d-tuple of commuting operators $T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$, there is a function $Q_T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ defined by $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$. It is easy to see that $Q_T^j(I) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} T^{*\alpha} T^\alpha \quad (j \geq 1)$ where $T^* = (T_1^*, \ldots, T_d^*)$. For each $m \geq 0$, let

$$P_m(T) = (I - Q_T)^m(I) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} Q_T^j(I).$$

A commuting tuple $T = (T_1, \ldots, T_d)$ is said to be a spherical m-isometry if $P_m(T) = 0$.

2. Main results

For a commuting d-tuple $T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$, the algebraic joint spectral radius defined by

$$r(T) := \inf_k \|Q_T^k(I)\|^{1/2k} = \lim_{k\to\infty} \|Q_T^k(I)\|^{1/2k}.$$

For $d = 1$, $r(T)$ is the usual spectral radius of T. The joint approximate point spectrum of T denoted by $\sigma_\pi(T)$ is defined by

$$\sigma_\pi(T) = \left\{ \begin{array}{r} \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \lim_{k\to\infty} \sum_{j=1}^{d} \|(T_j - \lambda_j)x_k\| = 0, \\ \text{for some sequence of unit vectors } \{x_k\}_k \end{array} \right\}$$

The d-tuple $T = (T_1, \ldots, T_d)$ is power regular if for each $x \in \mathcal{H}$,

$$r(x, T) := \lim_{k\to\infty} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^2 \right)^{1/2k} = \lim_{k\to\infty} \left\| \langle Q_T^k(I)x, x \rangle \right\|^{1/2k}$$

exists.

Theorem 2.1. *Let $T = (T_1, \ldots, T_d)$ be in $\mathcal{B}(\mathcal{H})^d$ and suppose that $x, y \in \mathcal{H}$ such that $r(x, T), r(y, T), r(x+y, T)$ exist. Then $r(x+y, T) \leq \max\{r(x, T), r(y, T)\}$. Moreover, if $T_i T_j = 0$, for $i \neq j$, then $r(x, T^k) = r(x, T)^k$ for all positive integers k, where $T^k = (T_1^k, \ldots, T_d^k)$.*

Theorem 2.2. *Every spherical m-isometry $T = (T_1, \ldots, T_d)$ is power regular. Moreover, for every non-zero vector $x \in \mathcal{H}$ the spectral radius of the restriction of the d-tuple T to the subspace*

$$M := \bigvee \{T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} x : n_1 \geq 0, n_2 \geq 0, \ldots n_d \geq 0\}$$

is one.

Proof. By Lemma 3.2 of [4], $\sigma_\pi(T)$ is in the boundary of the unit ball. Therefore, for every sequence of unit vectors $(x_k)_k$, there is j with $1 \leq j \leq d$ such that

$$\lim_{k\to\infty} T_j x_k \neq 0$$

and this implies that there is a positive constant c such that

$$|\langle Q_T(I)x, x\rangle| \geq c\|x\|^2 \quad \text{for all} \quad x \in \mathcal{H}. \tag{2.1}$$

To prove the existence of $r(x,T)$ note that $r(0,T) = 0$. So let x be a non-zero element in $\mathcal{H}$. Since T is a spherical m-isometry, then $P_j(T) = 0$ for $j \geq m$. Thus Lemma 2.2 of [4] implies that

$$\langle Q_T^k(I)x, x\rangle = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!}\langle P_j(T)x, x\rangle k^{(j)} \quad k = 1, 2, \cdots .$$

Note that $Q_T(I)$ is a positive operator. Moreover, $P_j(T)$'s are self-adjoint operators; therefore, $\langle Q_T^k(I)x, x\rangle$ is a polynomial in k with real coefficients of degree at most $m-1$ with non-negative leading coefficient

$$\frac{(-1)^{m-1}}{(m-1)!}\langle P_{m-1}(T)x, x\rangle.$$

Suppose that $\langle Q_T^k(I)x, x\rangle = a_0 + a_1 k + \ldots + a_{m-1}k^{m-1}$. Now, (2.1) implies that there exists $0 \leq i \leq m-1$ so that $a_i \neq 0$. Let a_r be the largest non-zero coefficient. Hence $\lim_{k\to\infty} |\langle Q_T^k(I)x, x\rangle|^{1/2k} = \lim_{k\to\infty} (a_r k^r)^{1/2k} = 1$. Therefore, T is power regular.

Moreover, for non-zero vector x in $\mathcal{H}$ we have

$$\begin{aligned}
1 &= r(x,T)\\
&= \lim_{k\to\infty}\left(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^2\right)^{1/2k}\\
&= \lim_{k\to\infty}\left(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha|_M x\|^2\right)^{1/2k}\\
&= \lim_{k\to\infty}\left(\left|\langle Q_{T|_M}^k(I)x, x\rangle\right|\right)^{1/2k}\\
&\leq \lim_{k\to\infty}\|Q_{T|_M}^k(I)\|^{1/2k}\|x\|^{1/k}\\
&= r(T|_M);
\end{aligned}$$

hence the spectral radius of the restriction of the tuple T to the subspace M is in the closed interval $[r(x,T), r(T)]$. Now since for every spherical m-isometry T, $r(T) = 1$ (see Proposition 3.1 of [4]) we get the result. $\square$

Corollary 2.3. *Every m-isometry is power regular. Moreover, for every non-zero vector $x \in \mathcal{H}$ the spectral radius of the restriction of the operator T to the subspace $M = \bigvee_{n\geq 0}\{T^n x\}$ is one.*

Recall that the d-tuple $T = (T_1, \ldots, T_d)$ is right invertible if there are operators $A_1, \ldots, A_d$ in $\mathcal{B}(\mathcal{H})$ such that $T_1A_1 + \cdots + T_dA_d = I$. Moreover, it is spherical unitary if T and T^* are spherical isometry. Since every spherical isometry is subnormal, T is a spherical unitary if and only if T is a normal spherical isometry. It is known that every spherical isometry on a finite-dimensional Hilbert space is necessarily a spherical unitary. On the other hand, there are examples of Taylor invertible spherical isometries which are not spherical unitaries [3, Theorem 3.1]. The question is under what conditions a Taylor invertible spherical isometry is a spherical unitary. In the next part of this section, we give sufficient conditions under which a right invertible spherical isometry is a spherical unitary.

Proposition 2.4. *Let $T = (T_1, \ldots, T_d)$ be a d-tuple of operators in $\mathcal{B}(\mathcal{H})$,*

(a) *If T is a spherical m-isometry then $Q_T(I)$ is invertible. Moreover, if $T_iT_j = 0$, for $i \neq j$ then $Q_{(T_1^{n_1},\ldots,T_d^{n_d})}(I)$ is invertible for every d-tuple $(n_1, \ldots, n_d)$ of positive integers. In particular, $Q_T^k(I)$ is invertible for every $k \geq 1$.*

(b) *If T is a right invertible spherical isometry such that $T_i^*T_j = 0$ for all $i \neq j$, then T is a spherical unitary.*

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A BASIC THEOREM FOR PROPER SPHERICAL m-ISOMETRIES

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Abstract. A commuting d-tuple $T = (T_1, \dots, T_d)$ is called a spherical m-isometry if $\sum_{j=0}^{m}(-1)^j\binom{m}{j}Q_T^j(I) = 0$, where $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$ for every bounded linear operator A on a Hilbert space $\mathcal{H}$. In this talk, we show that for every proper spherical m-isometry there are linearly independent operators $A_0, \dots, A_{m-1}$ such that $Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i$ for every $n \geq 0$.

1. Introduction

Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we set $|\alpha| = \sum_{j=1}^{d}\alpha_j$, $\alpha! = \alpha_1! \dots \alpha_d!$ and further $T^\alpha = T_1^{\alpha_1} \dots T_d^{\alpha_d}$. For every d-tuple of commuting operators $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$, there is a function $Q_T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ defined by $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$. It is easy to see that $Q_T^j(I) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} T^{*\alpha} T^\alpha \quad (j \geq 1)$ where $T^* = (T_1^*, \dots, T_d^*)$. For each $m \geq 0$, let

$$P_m(T) = (I - Q_T)^m(I) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}Q_T^j(I).$$

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* Speaker.

A commuting tuple $T = (T_1, \dots, T_d)$ is said to be a spherical m-isometry if $P_m(T) = 0$. For some information on spherical m-isometries one can see [1, 4] and [5].

2. Main results

Now we give a basic result about spherical m-isometries which is the multi-variable analog of Theorem 1 in [2]. For the proof of our main results, we need the following lemma.

Lemma 2.1. *If $P_m(k) = a_0 + a_1 k + \cdots + a_m k^m$, $m \geq 1$ then*

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} P_m(k) = m! a_m.$$

Theorem 2.2. *Let $\mathcal{H}$ be a Hilbert space. Then the d-tuple $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a proper spherical m-isometry if and only if there are $A_{m-1}, A_{m-2}, \dots, A_1, A_0$ in $\mathcal{B}(\mathcal{H})$ such that $A_{m-1} \neq 0$ and for every $n = 0, 1, 2, \dots$*

$$Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i.$$

Moreover, the sets $\{A_i : i = 0, \dots, m-1\}$ and $\{Q_T^{n-i}(A_i) : i = 0, \dots, m-1\}$ when $n \geq m$ are linearly independent.

Sketch of the proof. Assume that there are $A_{m-1} \neq 0, A_{m-2}, \dots, A_0$ so that

$$Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i, \quad \text{for all} \quad n \geq 0.$$

By applying Lemma 2.1 we observe that

$$P_m(T) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} Q_T^n(I) = 0.$$

Moreover,

$$\begin{aligned} P_{m-1}(T) = & \sum_{i=0}^{m-1} A_i \sum_{n=0}^{m-1} (-1)^n \binom{m-1}{n} n^i \\ = & (-1)^{m-1} (m-1)! A_{m-1} \\ \neq & 0. \end{aligned}$$

To prove the next part, suppose that $\sum_{i=0}^{m-1} x_i A_i = 0$ for some scalers $x_1, \dots, x_{m-1}$. Therefore

$$0 = \sum_{i=0}^{m-1} x_i A_i = \sum_{j=0}^{m-1} \left[\frac{(-1)^j}{j!} \sum_{i=0}^{j} x_i \alpha_{(i,j)} \right] P_j(T).$$

Now it is sufficient to show that $\{P_j(T) : j = 0, \dots, m-1\}$ is a linearly independent set. Indeed, in this case $\sum_{i=0}^{j} x_i \alpha_{(i,j)} = 0$ for $j = 0, \dots, m-1$, and since the matrix of the coefficients of this system is lower triangular with diagonal components $\alpha_{(j,j)} = 1, \quad j = 0, \dots, m-1$, we get $x_i = 0$ for $i = 0, \dots, m-1$. To finish the proof of this part suppose that $\sum_{k=0}^{m-1} \alpha_k P_k(T) = 0$ for some complex numbers $\alpha_1, \dots, \alpha_{m-1}$. Then $\sum_{k=0}^{m-1} \alpha_k Q_T(P_k(T)) = 0$. On the other hand,

$$\sum_{k=0}^{m-1} \alpha_k P_{k+1}(T) = 0.$$

By continuing this way we get

$$\sum_{k=0}^{m-1} \alpha_k P_k(T) = \sum_{k=0}^{m-1} \alpha_k P_{k+1}(T) = \cdots = \sum_{k=0}^{m-1} \alpha_k P_{k+m-1}(T) = 0.$$

Moreover, since $P_{m-1}(T) \neq 0$ and

$$0 = \sum_{k=0}^{m-1} \alpha_k P_{k+m-1}(T) = \alpha_0 P_{m-1}(T),$$

we conclude that $\alpha_0 = 0$. In the next step we have

$$0 = \sum_{k=0}^{m-1} \alpha_k P_{k+m-2}(T) = \alpha_1 P_{m-1}(T);$$

thus $\alpha_1 = 0$. Continuing this process we obtain $\alpha_k = 0$ for all $k = 0, 1, \dots, m-1$.

To prove the last part of the theorem, suppose that there are scalars $x_1, \dots, x_{m-1}$ such that

$$0 = \sum_{i=0}^{m-1} x_i Q_T^{n-i}(A_i) = \sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} Q_T^{n-i}(P_j(T)).$$

As T is a proper spherical m-isometry, there is $x \in \mathcal{H}$ such that $P_{m-1}(T)x \neq 0$. On the other hand, one can see

$$\sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \left\langle Q_T^{n-i}(P_{j+1}(T))x, P_{m-1}(T)x \right\rangle = 0.$$

Applying this process, we will have

$$\sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \left\langle Q_T^{n-i} P_{j+k}(T)x, P_{m-1}(T)x \right\rangle = 0$$

for all $0 \leq k \leq m-1$. Take $k = m-1$ we have

$$Q_T^n(P_{m-1}(T)) = P_{m-1}(T)$$

for all $n \geq m$; thus $x_0 = 0$. In the next step taking $k = m-2$ we get

$$x_1 \alpha_{(1,1)} \|P_{m-1}(T)x\|^2 = 0$$

then $x_1 = 0$. Continuing this process we will have $x_i = 0$ for all $i = 0, \ldots, m-1$.

The reverse implication is easy to prove. □

Proposition 2.3. *Let $T = (T_1, \ldots, T_d)$ be a proper spherical m-isometry. Then the following sets are linearly independent.*

(1) $\{P_k(T) : k = 0, 1, \ldots, m-1\}$.
(2) $\{Q_T^k(I) : k = 0, 1, \ldots, m-1\}$.
(3) $\{Q_T^{n-k}(P_k(T)) : k = 0, 1, \ldots, m-1\}$ *when* $n \geq m$.

Corollary 2.4 (Theorem 3.1 in [3])**.** *If $A \in \mathcal{B}(\mathcal{H})$ is a proper m-isometry then the set $\{I, A^*A, \ldots, A^{*m-1}A^{m-1}\}$ is linearly independent.*

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A REMARK ON DISCONTINUITY AT FIXED POINT OVER PARTIAL METRIC SPACES

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Abstract. In this article, we prove the existence of a fixed point for some contractive mappings over partial metric spaces. We shall show that this kind of mappings are not continuous in their fixed points.

1. Introduction

Partial metric spaces, which are a generalization of metric spaces, introduced by S. G. Mathews [3] as a part of the study of denotational semantics of data flow networks. He gave a Banach fixed point result for these spaces. After that, many authors proved fixed point theorems on partial metric spaces. In 1999, Pant [4] proved the following fixed point theorem and obtained the first result that intromit discontinuity at the fixed point. For more details see [1, 2, 5].

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* Speaker.

Theorem 1.1. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions:*

(i) $d(Tx, Ty) \leq \phi(\max\{d(x, Tx), d(y, Ty)\})$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$.

(ii) *For a given* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that* $\epsilon < \max\{d(x, Tx), d(y, Ty)\} < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$,

then T has a unique fixed point, say z. Moreover, T is continuous at z iff $\lim_{x\to z} d(x, Tx) = 0$.

Definition 1.2 ([3]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $p(x, x) = p(y, y) = p(x, y) \iff x = y$.
(p2) $p(x, x) \leq p(x, y)$.
(p3) $p(x, y) = p(y, x)$.
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

By (P1) and (P2), it is easy to see that $p(x, y) = 0$ implies that $x = y$. A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is obvious that any metric is a partial metric.

Example 1.3. (1) Let $X = (-\infty, 0]$. We consider the function $p : X \times X \longrightarrow \mathbb{R}^+$ defined by $p(x, y) = -\min\{x, y\}$ for any $x, y \in X$. Then p is a partial metric on X.

(2) Let $p : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be defined by $p(x, y) = \max\{x, y\}$ for any $x, y \in X$. Then p ia a partial metric on $\mathbb{R}^+$.

Any partial metric p on X induces a topology on (X, p) which has a base of p-balls:

$$B_p(x, \epsilon) = \{y \in X : \ p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$. We denote this topology by τ_p. The sequence $(x_n)_n$ in X is called Cauchy if $\lim\limits_{m,n\to\infty} p(x_n, x_m)$ exists and is finite. This sequence is convergent to x if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_n, x_m).$$

The space (X, p) is called complete if every Cauchy sequence in X converges.

A mapping $f : X \longrightarrow Y$ between two partial metric spaces (X, p) and (Y, q) is called τ_p-continuous (or simply continuous) at $x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(B_p(x_0, \delta)) \subseteq B_q(fx_0, \epsilon).$$

Recall that f is sequentially continuous in x_0 if $fx_n \to fx_0$ whenever $x_n \to x_0$.

Theorem 1.4. *Suppose that $f : X \longrightarrow Y$ is a map between partial metric spaces (X, p) and (Y, q). Then f is continuous if and only if it is sequentially continuous.*

2. Main results

In what follows we shall denote

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), [p(x, Ty) + p(y, Tx)]/2\}. \tag{2.1}$$

Theorem 2.1. *Let (X, p) be a complete partial metric space. Let T be a self-mapping on X such that T^2 is continuous and satisfy the conditions:*

(i) *$p(Tx, Ty) \leq \phi(M(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\phi(t) < t$ for each $t > 0$.*
(ii) *For a given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies that $p(Tx, Ty) \leq \delta$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z iff $\lim_{x \to z} M(x, z) \neq 0$.

The following example illustrates the above theorem.

Example 2.2. Let $X = [0, 2]$ and p be the usual metric on X. Define $T : X \to X$ by $T(x) = 1$ for $x \leq 1$ and $T(x) = 0$ for $x > 1$. Then T satisfies the conditions of Theorem 2.1 and has a unique fixed point $x = 1$

Corollary 2.3. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous and satisfies the conditions:*

(i) *$d(Tx, Ty) < M(x, y)$, for every $x, y \in X$ with $M(x, y) > 0$.*
(ii) *For a given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies that $d(Tx, Ty) \leq \epsilon$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z iff $\lim_{x \to z} M(x, z) \neq 0$.

Corollary 2.4. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous and satisfies the conditions:*

(i) *$d(Tx, Ty) \leq \phi[d(x, y)]$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\phi[d(x, y)] < d(x, y)$.*

(ii) *For for a given* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for any* $t > 0$, $\epsilon < t < \epsilon + \delta$ *implies that* $\phi(t) \leq \epsilon$.

Then T *has a unique fixed point, say* z, *and* $T^n x \to z$ *for each* $x \in X$.

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ON RECENT DEVELOPMENTS IN OPERATOR ENTROPY

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Abstract. In this paper, we review the recent developments of operator entropy and reverses of it.

1. Introduction and preliminaries

Throughout the paper, let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and I be the identity operator. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$ and denote its identity by I_n. A self-adjoint operator $A \in \mathbb{B}(\mathcal{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and then we write $A \geq 0$. An operator A is said to be *strictly positive* (denoted by $A > 0$) if it is positive and invertible. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Let f be a continuous real valued function defined on an interval J. The function f is called *operator monotone* if $A \leq B$ implies $f(A) \leq f(B)$ for all $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in J. The function f is said to be *operator*

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concave on J if

$$\lambda f(A)+(1-\lambda)f(B)\leq f(\lambda A+(1-\lambda)B)$$

for all self-adjoint operators $A,B\in\mathbb{B}(\mathcal{H})$ with spectra in J and all $\lambda\in[0,1]$. Also, we say that $A\in\mathbb{B}(\mathcal{H})$ is *contraction* if $A^*A\leq I$.

2. Operator Entropy Inequalities

In this section, we present a type of entropy for operators. Then we will conclude a very important inequality for the operator entropy. The object of this section is to state an operator entropy inequality parallel to the main result of [1].

A relative operator entropy of strictly positive operators A,B was introduced in the noncommutative information theory by Fujii and Kamei [2] and is defined by

$$S(A|B)=A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}\,.$$

For positive operators A,B, one may set $S(A|B):=s\text{-}\lim_{\epsilon\to+0}S(A+\epsilon 1_{\mathcal{H}}|B)$ if it exists.

Definition 2.1 ([4]). Assume $\mathbf{A}=(A_1,\cdots,A_n)$ and $\mathbf{B}=(B_1,\cdots,B_n)$ are finite sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. For $q\in\mathbb{R}$ and an operator monotone function $f:(0,\infty)\to[0,\infty)$ the *generalized operator Shannon entropy* is defined by

$$S_q^f(\mathbf{A}|\mathbf{B}):=\sum_{j=1}^{n}S_q^f(A_j|B_j)\,, \tag{2.1}$$

where $S_q^f(A_j|B_j)=A_j^{1/2}\left(A_j^{-1/2}B_jA_j^{-1/2}\right)^q f\left(A_j^{-1/2}B_jA_j^{-1/2}\right)A_j^{1/2}$.

We recall that for $q=0$, $f(t)=\log t$ and $A,B>0$, we get the relative operator entropy $S_0^f(A|B)=A^{\frac{1}{2}}\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}=S(A|B)$. It is interesting to point out that $S_q(A|B)=-S_{1-q}(B|A)$ for every real number q, in particular, $S_1(A|B)=-S(B|A)$.

The following result gives lower and upper bounds for $S_q^f(\mathbf{A}|\mathbf{B})$.

Theorem 2.2 ([4]). *Assume that f, $\mathbf{A}$ and $\mathbf{B}$ are as in Definition 2.1. If $\sum_{j=1}^n A_j=\sum_{j=1}^n B_j=I_{\mathcal{H}}$ and f is operator concave, then*

$$\begin{aligned}&f\left[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j)+t_0\left(I_{\mathcal{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\right]-f(t_0)\left(I_{\mathcal{H}}-\sum_{j=1}^{n}A_j\natural_pB_j\right)\\&\geq S_p^f(\mathbf{A}|\mathbf{B})\end{aligned} \tag{2.2}$$

for all $p \in [0,1]$ and for any fixed real number $t_0 > 0$, and

$$-f\left[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\left(I_{\mathcal{H}} - \sum_{j=1}^{n} A_j \natural_p B_j\right)\right] + f(t_0)\left(I_{\mathcal{H}} - \sum_{j=1}^{n} A_j \natural_p B_j\right)$$
$$\leq S_p^f(\mathbf{A}|\mathbf{B}) \tag{2.3}$$

for all $p \in [2,3]$ and for any fixed real number $t_0 > 0$, where $X \natural_q Y$ is defined by $X^{\frac{1}{2}}\left(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}}\right)^q X^{\frac{1}{2}}$ for any real number q and any strictly positive operators X and Y. For $p \in [0,1]$, the operator $X \natural_p Y$ coincides with the well-known p-power mean of X, Y.

Remark 2.3. By taking $f(t) = \log t$ in Theorem 2.6, we get Theorem 2.1 of [1].

Corollary 2.4 (Operator Entropy Inequality, [4]). *Assume that $A_1, \cdots, A_n \in \mathbb{B}(\mathcal{H})$ are positive invertible operators satisfying $\sum_{j=1}^{n} A_j = I_{\mathcal{H}}$. Then*

$$-\sum_{j=1}^{n} A_j \log A_j \leq (\log n) I_{\mathcal{H}}\,.$$

Let $\mathcal{A}$ be a C^*-algebra of operators acting on a Hilbert space, let T be a locally compact Hausdorff space and $\mu(t)$ be a Radon measure on T. A field $(A_t)_{t\in T}$ of operators in $\mathcal{A}$ is called a *continuous field of operators* if the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathcal{A}$ such that

$$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t) \tag{2.4}$$

for every linear functional φ in the norm dual $\mathcal{A}^*$ of $\mathcal{A}$; see [3].

Suppose that $\mathbf{A} = (A_t)_{t\in T}, \mathbf{B} = (B_t)_{t\in T}$ are (continuous) fields of strictly positive operators, $q \in \mathbb{R}$ and f is a nonnegative operator monotone function on $(0,\infty)$. Then we have the definition of the generalized relative operator entropy

$$\widetilde{S}_q^f(\mathbf{A}|\mathbf{B}) := \int_T S_q^f(A_s|B_s) d\mu(s)\,, \tag{2.5}$$

where $S_q^f(A_s|B_s) = A_s^{1/2}\left(A_s^{-1/2} B_s A_s^{-1/2}\right)^q f\left(A_s^{-1/2} B_s A_s^{-1/2}\right) A_s^{1/2}$. In the next theorem we have an extension of (2.1).

Theorem 2.5. *Let $\mathbf{A} = (A_t)_{t\in T}, \mathbf{B} = (B_t)_{t\in T}$ be continuous fields of strictly positive operators such that $0 < mA_s \leq B_s \leq MA_s$ ($s \in T$) for some positive real numbers m, M, where $m < 1 < M$, and*

$\int_T A_s d\mu(s) = \int_T B_s d\mu(s) = I_{\mathcal{H}}$, $f : (0,\infty) \to [0,\infty)$ *be operator concave and* $p \in [0,1]$. *Then*

$$\begin{aligned} f&\left[\int_T (A_s \natural_{p+1} B_s) d\mu(s) + t_0\left(I_{\mathcal{H}} - \int_T A_s \natural_p B_s d\mu(s)\right)\right] \\ &- f(t_0)\left(I_{\mathcal{H}} - \int_T A_s \natural_p B_s d\mu(s)\right) \\ \geq& \widetilde{S}_p^f(\mathbf{A}|\mathbf{B}). \end{aligned} \tag{2.6}$$

If f is a strictly concave differentiable function on an interval $[m, M]$ with $m < M$ and $\Phi : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{K})$ is a positive unital linear map,

$$\mu_f = \frac{f(M) - f(m)}{M - m}, \ \nu_f = \frac{Mf(m) - mf(M)}{M - m} \tag{2.7}$$

$$\text{and} \quad \gamma_f = \max\left\{\frac{f(t)}{\mu_f t + \nu_f} : m \leq t \leq M\right\}, \tag{2.8}$$

then

$$f(\Phi(A)) \leq \gamma_f \Phi(f(A)). \tag{2.9}$$

Theorem 2.6. *Let* $\mathbf{A} = (A_t)_{t\in T}, \mathbf{B} = (B_t)_{t\in T}$ *be continuous fields of strictly positive operators such that* $0 < mA_s \leq B_s \leq MA_s$ $(s \in T)$ *for some positive real numbers* m, M, *where* $m < 1 < M$, $\int_T A_s d\mu(s) = \int_T B_s d\mu(s) = I_{\mathcal{H}}$, $f : (0,\infty) \to [0,\infty)$ *be operator concave and* $p \in [0,1]$. *Then*

$$f\left[\int_T (A_s \natural_{p+1} B_s) d\mu(s) + t_0\left(I_{\mathcal{H}} - \int_T A_s \natural_p B_s d\mu(s)\right)\right] \tag{2.10}$$

$$\begin{aligned} &- \gamma_f f(t_0)\left(I_{\mathcal{H}} - \int_T A_s \natural_p B_s d\mu(s)\right) \\ \leq& \gamma_f \widetilde{S}_p^f(\mathbf{A}|\mathbf{B}), \end{aligned} \tag{2.11}$$

where $t_0 \in [m, M]$ *and* γ_f *is given by* (2.7).

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ON A CONJECTURE ABOUT A NONLINEAR MATRIX EQUATION

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Abstract. Let A be a positive definite matrix and B a positive semidefinite matrix. In this paper, we discuss on a conjecture about operator equation $f(A)X + Xf(A) = AB + BA$.

1. Introduction

It is known that $A, B \geq 0$ does not imply that $AB + BA \geq 0$. In [1], Chan and Kwong, studying some inequalities about $AB + BA$, proved the following results:

Lemma 1.1. *Let A be a positive definite matrix and Q a nonnegative Hermitian matrix. The solution X of the following matrix equation is always positive semidefinite.*

$$A^2X + XA^2 = Q. \tag{1.1}$$

Theorem 1.2. *Let A be a positive definite matrix and B a positive semidefinite matrix. The solution X of the following matrix equation is always positive semidefinite.*

$$A^2X + XA^2 = AB + BA. \tag{1.2}$$

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At the end of the paper, the following question was posed associated with above theorem:
Question: How can one characterize all the functions f such that the solution of the matrix equation

$$f(A)X + Xf(A) = AB + BA \tag{1.3}$$

is positive semidefinite?

Our aim in this note is to seek a soloution for this question.

2. RESULTS

We recall the Furuta's inequality.

Theorem 2.1 ([2]). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$.

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$,

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Using Theorem 2.1, Furuta showed the existence of positive semidefinite solution of the following operator equation in Hilbert space (related to 1.2 and 1.3) via an order preserving operator inequality.

$$\sum_{j=1}^{n} A_{n-j}XA_{j-1} = B,$$

where A is a positive definite operator and B is a self-adjoint operator([3]).

Using the same method in [1], we can prove the following lemma.

Lemma 2.2. *Let A be a positive definite matrix and Q a nonnegative Hermitian matrix. If f is a nonnegative continuous function on $(0,\infty)$, then the solution of the following equation is always positive semi-definite.*

$$f(A)X + Xf(A) = Q. \tag{2.1}$$

Proof. Set $Y(t) = (f^2(A) + tQ)^{\frac{1}{2}}$. Since $f^2(A) + tQ \geq f^2(A)$ for any $t \geq 0$, $Y^2(t) \geq Y^2(0)$ which implies that $Y(t) \geq Y(0)$. Hence, $Y'(0) \geq 0$.

On the other hand, by differentiating the equation $Y^2(t) = f^2(A) + tQ$ and then letting $t = 0$,

$$Y(0)Y'(0) + Y'(0)Y(0) = \frac{d}{dt}(f^2(A) + tQ)|_{t=0}.$$

Hence, we have for $X = Y'(0) \geq 0$,

$$f(A)X + Xf(A) = Q.$$

□

Definition 2.3. Let A, B be positive definite matrices. We say that A and B are monotone sub-additive matrices if for all nonnegative matrix monotone functions g on $[0, \infty)$, we have

$$g(A + B) \leq g(A) + g(B).$$

Moslehian and Najafi proved in [4] that A and B are monotone sub-additive if and only if $AB + BA$ is positive definite. Thus we will have the following corollary.

Corollary 2.4. *Let A, B be positive definite matrices. Suppose that f is a nonnegative continuous function on $[0, \infty)$. If A and B are monotone sub-additive, then the solution of the following equation is always positive semidefinite.*

$$f(A)X + Xf(A) = BA + AB. \tag{2.2}$$

Proof. Since A and B are monotone sub-additive, so $AB + BA$ is positive definite. By Theorem 2.2, equation (2.2) has positive definite solution for any non-negative function f. □

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ON APPROXIMATE PARALLELISM

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Abstract. We introduce a new definition of ε-parallelism in normed spaces and we try to explain some properties of it in some special normed spaces such as the space of operators on a Hilbert space.

1. Introduction

Let $\mathcal{X}$ be a complex normed space. We try here to introduce a type of approximate parallelism in an arbitrary normed space. Parallelism in an inner product space means the linearly dependence. In an arbitrary normed space $\mathcal{X}$, a type of parallelism is considered in [4] and [2] as follows:

$$x||y \iff \exists\lambda \in \mathbb{T},\ \|x+\lambda y\| = \|x\| + \|y\|$$

where $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$. In [2], the authors gave an example to show that the parallelism does not imply the linear dependence (Example 2.2 in the paper). The same paper also rises a type of approximate parallelism as follows:

$$x||^{\varepsilon}y \iff \inf\{\|x+\mu y\| : \mu \in \mathbb{C}\} \leq \varepsilon\|x\|$$

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and it is stated that in the case when $\varepsilon = 0$ this definition implies the definition of exact parallelism where they show 0-parallelism implies a linear dependence. But this and above mentioned example show that the 0-parallelism implies the exact parallelism introduced in that paper but not necessarily the converse.

Inspired by the definition of approximate parallelism in [2], a definition of ε-parallelism is given in Section 3 which in the case mentioned above it coincides with the exact parallelism. Utilizing the results of [2] for Birkhoff-James-orthogonality, we present some similar results for approximate Birkhoff-James-orthogonality in square matrix spaces, operator spaces, C^*-algebras and Hilbert C^*-modules.

The notion of Hilbert C^*-module is a generalization of Hilbert space which the role of complex scalers are played by elements of a C^*-algebra $\mathcal{A}$. An inner product $\mathcal{A}$-module is a complex linear space $\mathcal{E}$ which is a right $\mathcal{A}$-module with a compatible scaler multiplication which equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying

(1) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$.
(2) $\langle x, ya \rangle = \langle x, y \rangle a$.
(3) $\langle x, y \rangle^* = \langle y, x \rangle$.
(4) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,

for all $x, y, z \in \mathcal{E}, a \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. Note that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{E}$ due to the Cauchy-Schwartz inequality. If $\mathcal{E}$ is complete with respect to this norm, then it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-algebra over $\mathcal{A}$. for every $x \in \mathcal{E}$ the positive square root of $\langle x, x \rangle$ is denoted by $|x|$. Let φ be a state over $\mathcal{A}$, which is a positive linear functional on $\mathcal{A}$ of norm one, then

$$|\varphi(\langle x, y \rangle)|^2 \leq \varphi(\langle x, x \rangle)\varphi(\langle y, y \rangle).$$

The interested reader is referred to [1] for further information on Hilbert C^*-modules.

2. ε-PARALLELISM

Definition 2.1. Let $(\mathcal{X}, \|.\|)$ be a normed space and x and y be two non-zero elements of $\mathcal{X}$. The vector x is ε-parallelism to y, denoted by $x||^{\varepsilon}y$, if there exists a real number θ such that $(\frac{x}{\|x\|} + \varepsilon e^{i\theta}\frac{y}{\|y\|})||y$, i.e., there is a $\lambda \in \mathbb{T}$ such that

$$\left\| \frac{x}{\|x\|} + \frac{\varepsilon}{2} e^{i\theta} \frac{y}{\|y\|} + \lambda y \right\| = \left\| \frac{x}{\|x\|} + \frac{\varepsilon}{2} e^{i\theta} \frac{y}{\|y\|} \right\| + \|y\|.$$

Obviously when $\varepsilon = 0$, ε-parallelism is the same as parallelism. Let $(\mathcal{X}, \langle .,. \rangle)$ be an inner product space and x and y be in $\mathcal{X}$. It is easy to see that $x||y$ if and only if $|\langle x,y\rangle| = \|x\|\|y\|$. This fact suggests the following ε-parallelism in inner product spaces.

Definition 2.2. In an inner product space $\mathcal{X}$ for $\varepsilon \in [0,1)$, we say that x is ε-i-parallel to y, denoted by $x||_i^\varepsilon y$, if

$$|\langle x,y\rangle| \geq (1-\varepsilon)\|x\|\|y\|.$$

Theorem 2.3. *Let $(\mathcal{X}, \langle .,. \rangle)$ be an inner product space and x and y be non-zero elements in $\mathcal{X}$ and $\varepsilon \in [0,1)$ such that $x||^\varepsilon y$. Then $x||_i^\varepsilon y$.*

Proof. $x||^\varepsilon y$ implies that $\frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}||y$ for some real number θ. Thus

$$\begin{aligned}\frac{1}{\|x\|}|\langle x,y\rangle| + \frac{\varepsilon}{2}\|y\| &\geq |\langle \frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}, y\rangle| \\ &= \|\frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}\|\|y\| \geq (1-\frac{\varepsilon}{2})\|y\|.\end{aligned}$$

Therefore

$$|\langle x,y\rangle| \geq (1-\varepsilon)\|y\|\|x\|.$$

□

This theorem is the reason why we provide $\frac{\varepsilon}{2}$ in the Definition 2.1 instead ε.

Now we want to pay our attention to the ε-parallelism of operators on a Hilbert space $\mathcal{H}$.

Theorem 2.4. *Let T_1 and T_2 be two non-zero operators in $\mathbb{B}(\mathcal{H})$. The following statements are equivalent:*

(a) $T_1||^\varepsilon T_2$.
(b) $T_1^*||^\varepsilon T_2^*$.
(c) $\alpha T_1||^\varepsilon \beta T_2$ *for* $\alpha, \beta \in \mathbb{R}\setminus\{0\}$.
(d) $\gamma T_1||^\varepsilon \gamma T_2$ *for all* $\gamma \in \mathbb{C}\setminus\{0\}$.
(e) *There exists a sequence of unite vectors $\{\xi_n\}$ in $\mathcal{H}$ and $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{T}$ such that*

$$\lim_{n\to\infty}\left\langle \frac{T_1\xi_n}{\|T_1\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{T_2\xi_n}{\|T_2\|}, T_2\xi_n \right\rangle = \lambda \left\| \frac{T_1}{\|T_1\|} + \frac{\varepsilon}{2}\frac{T_2}{\|T_2\|}\right\| \|T_2\|.$$

(f) *For some* $\theta \in \mathbb{R}$

$$\begin{aligned} r\left(T_2^*\left(\frac{T_1}{\|T_1\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{T_2}{\|T_2\|}\right)\right) &= \left\|T_2^*\left(\frac{T_1}{\|T_1\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{T_2}{\|T_2\|}\right)\right\| \\ &= \left\|\frac{T_1}{\|T_1\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{T_2}{\|T_2\|}\right\| \|T_2\|.\end{aligned}$$

Corollary 2.5. *Let T_1 and T_2 be two non-zero operators in $\mathbb{B}(\mathcal{H})$ such that $T_1||^\varepsilon T_2$. Then*

(i) *There exists a sequence of unite vectors* $\{\xi_n\}$ *in* $\mathcal{H}$ *such that*

$$\lim_{n\to\infty} |\langle T_1\xi_n, T_2\xi_n\rangle| \geq (1-\varepsilon)\|T_1\|\|T_2\|.$$

(ii) $\|T_1^*T_2\| = \|T_2^*T_1\| \geq (1-\varepsilon)\|T_1\|\|T_2\|$.

Theorem 2.6. *Let* $\mathcal{E}$ *be a Hilbert* C^**-modules over* $\mathcal{A}$*. For* $x, y \in \mathcal{E}$ *the following are equivalent:*

(a) $x||^{\varepsilon}y$.

(b) *There exist a state* φ *on* $\mathcal{A}$ *and a real number* θ *such that*

$$\varphi(\langle \frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}, y\rangle) \;=\; \lambda\left\|\frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}\right\|\|y\|. \tag{2.1}$$

(c) *There is a linear functional* f *of norm one on* $\mathcal{E}$*,* $\lambda \in \mathbb{T}$ *and real number* θ *such that* $f(\frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}) = \|\frac{x}{\|x\|} + \frac{\varepsilon}{2}e^{i\theta}\frac{y}{\|y\|}\|$ *and* $f(y) = \lambda\|y\|$.

Corollary 2.7. *Let* $\mathcal{E}$ *be a Hilbert* C^**-modules over* $\mathcal{A}$ *and let* $x, y \in \mathcal{E}$ *such that* $x||^{\varepsilon}y$*. Then there exist a state* φ *on* $\mathcal{A}$ *and a real number* θ *such that*

$$|\varphi(\langle x, y\rangle)| \geq (1-\varepsilon)\|x\|\|y\|.$$

Recall that the C^*-algebra $\mathcal{A}$ is a Hilbert C^*-module over itself, in which inner product $\mathcal{A}$-module is defined by $\langle a, b\rangle = a^*b$ for $a, b \in \mathcal{A}$.

Corollary 2.8. *Let* $\mathcal{A}$ *be a* C^**-algebra and* $a, b \in \mathcal{A}$ *such that* $a||^{\varepsilon}b$*. Then there exists a state* φ *on* $\mathcal{A}$ *such that*

$$|\varphi(a^*b)| \geq (1-\varepsilon)\|a\|\|b\|.$$

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MULTIPLE SOLUTIONS FOR A CLASS OF NONLOCAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. Using variational techniques, we study multiplicity of non-negative nontrivial weak solution for a class of nonlocal equations. The proofs rely essentially on the minimum principle combined with the mountain pass theorem.

1. Introduction

In this paper, we are concerned with the following Kirchhoff type problem:

$$\begin{cases} -M\Big(\int_\Omega H(|\nabla u|^p)dx\Big)\ \operatorname{div}\Big(h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\Big)=\lambda f(x,u) & \text{in } \Omega, \\ -M\Big(\int_\Omega H(|\nabla u|^p)dx\Big)\ h(|\nabla u|^p)|\nabla u|^{p-2}\dfrac{\partial u}{\partial n}=\mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $R^N(N \geq 3)$, $\dfrac{\partial u}{\partial n}$ is the outer unit normal derivative, $2 \leq p, q < N$, h is an increasing continuous function from R into R, λ, μ are parameters, $H(t) = \int_0^t h(s)ds$ for all $t \in R$ and $M : R^+ \to R$ is a continuous function, $f : \Omega \times R \to R, g : \partial\Omega \times R \to R$ satisfy the Caratheodory condition.

In [1] , the authors studied the existence and nonexistence of nontrivial weak solution for a class of general Capillarity systems. We point out the fact that if $h \equiv 1$, problem (1.1) becomes a nonlocal Kirchhoff type equation with nonlinear boundary condition which extends d'Alembert wave equation with free vibrations of elastic strings [3].

In order to state the main result, we need the following assumptions for f and g. Denote $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$, then we assume that

(**F1**) $f : \Omega \times R \to R$ is a Caratheodory function such that

$$|f(x,t)| \leq C_1(1 + |t|^{p-1}) \qquad ((x,t) \in \Omega \times R),$$

(**G1**) $g : \partial\Omega \times R \to R$ satisfies the Caratheodory function such that

$$|g(x,t)| \leq C_2(1 + |t|^{p-1}) \qquad ((x,t) \in \partial\Omega \times R).$$

We say that a function γ verifies the property (Γ) if and only if

$$\gamma(t) \leq C|t|^p \tag{Γ}$$

for all $t \in R$, where $C > 0$ is independent of γ. Let K_i, $i = 1, ..., 4$, be four functions satisfying property Γ. We introduce the following assumptions on the behavior of F and G at origin and at infinity:

(**F2**) It holds that

$$\limsup_{t\to 0} \frac{F(x,t)}{K_1(t)} \leq 0$$

uniformly in $x \in \Omega$;

(**F3**) It holds that

$$\limsup_{t\to \infty} \frac{F(x,t)}{K_2(t)} \leq 0$$

uniformly in $x \in \Omega$.

(**G2**) It holds that

$$\limsup_{t\to 0} \frac{G(x,t)}{K_3(t)} \leq 0$$

uniformly in $x \in \partial\Omega$;

(**G3**) It holds that
$$\limsup_{t\to\infty}\frac{G(x,t)}{K_4(t)}\le 0$$
uniformly in $x\in\partial\Omega$.

Regarding the functions h, M, we assume that

(**M1**) There are two positive constants m_0, m_1 such that
$$m_0\le M(t)\le m_1 \qquad (t\ge 0).$$

(**H1**) $h:[0,+\infty)\to R$ is an increasing continuous function and there exist α, $\beta>0$ such that
$$\alpha\le h(t)\le\beta \qquad (t\ge 0).$$

(**H2**) There is constants $\gamma>0$ such that
$$\Big(h(|\xi|^p)|\xi|^{p-2}\xi-h(|\eta|^p)|\eta|^{p-2}\eta\Big)\cdot(\xi-\eta)\ge\gamma|\xi-\eta|^p,$$
for all $\xi,\eta\in R^N$.

Let $X=W_0^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm
$$\|u\|_{1,p}^p=\int_\Omega|\nabla u|^p dx,$$
Let us define the functionals $\rho: X\to R$ by
$$\rho(u)=\int_\Omega H(|\nabla u|^p)dx, \tag{1.2}$$
and the mapping $J:X\to R$ by
$$J(u)=\frac{1}{p}\widehat{M}\Big(\rho(u)\Big)$$
where
$$\widehat{M}(t)=\int_0^t M(s)ds.$$
Also, we define the mapping $I:X\to R$ and $\psi:X\to R$ by
$$I(u)=\int_\Omega F(x,u)dx, \qquad \psi(u)=\int_{\partial\Omega}G(x,u)d\sigma \tag{1.3}$$
The main results of this paper are as follows.

Theorem 1.1. *Suppose that* (**F1**)-(**F3**), (**G1**)-(**G3**), (**M1**) *and* (**H1**)-(**H2**) *are satisfied. Moreover, we assume that there exist t_0, such that $F(x,t_0)>0$ for all $x\in\Omega$. Then, there exist $\lambda^*,\mu^*>0$ such that system (1.1) has at least two distinct, nonnegative, nontrivial weak solutions, provided that $\lambda\ge\lambda^*$ and $\mu\ge\mu^*$.*

2. Proofs of the main results

We will prove Theorem 1.1, using critical point theory. Set $f(x,t) = g(x,t) = 0$ for $t < 0$. For all $\lambda, \mu \in R$, we consider the functional $E_{\lambda,\mu}(u) : X \to R$ given by

$$E_{\lambda,\mu} = J(u) - \lambda I(u) - \mu\psi(u). \tag{2.1}$$

By (**F1**), (**G1**), a simple computation implies that $E_{\lambda,\mu}$ is well-defined and of C^1 class in X. Thus, weak solutions of system (1.1) correspond to the critical points of $E_{\lambda,\mu}$.

By applying the minimum principle in [4] we can prove that $E_{\lambda,\mu}$ has a global minimizer u_1 which is a non-negative, non-trivial weak solution of problem (1.1)

Lemma 2.1. *There exist a constant $\rho \in (0, \|u_1\|_X)$ and a constant $r > 0$ such that $E_{\lambda,\mu}(u) \geq r$ for all $u \in X$ with $\|u\|_X = \rho$.*

Lemma 2.2. *The functional $E_{\lambda,\mu}$ satisfies the Palais-Smale condition in X.*

Proof of Theorem 1.1. By applying the minimum principle, system (1.1) admits a non-negative, non-trivial weak solution u_1 as the global minimizer of $E_{\lambda,\mu}$. Setting

$$\overline{c} := \inf_{\chi\in\mathbf{\Gamma}} \max_{u\in\chi([0,1])} E_{\lambda,\mu}(u), \tag{2.2}$$

where $\mathbf{\Gamma} := \{\chi \in C([0,1], X) : \chi(0) = 0, \chi(1) = u_1\}$.

Lemmas 2.1- 2.2 show that all assumptions of the mountain pass theorem in [2] are satisfied, $E_{\lambda,\mu}(u_1) < 0$ and $\|u_1\|_X > \rho$. Then, $\overline{c}$ is a critical value of $E_{\lambda,\mu}$, i.e. there exists a weak solution $u_2 \in X$ of (1.1). Moreover, u_2 is not trivial and $u_2 \not\equiv u_1$ since $E_{\lambda,\mu}(u_2) = \overline{c} > 0 > E_{\lambda,\mu}(u_1)$. Thus Theorem 1.1 is completely proved. □

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ANALYSIS OF THE STABILITY AND CONVERGENCE OF A FINITE DIFFERENCE APPROXIMATION FOR STOCHASTIC BURGER'S EQUATION

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Abstract. In this paper, an implicit finite difference scheme is proposed for the numerical solution of stochastic Burger's equation of Itô type. The consistency, stability and convergence of the scheme is analyzed.

1. Introduction

Stochastic partial differential equations (SPDEs) play a prominent role in a range of applications, including biology, chemistry, epidemiology, mechanics, microelectronics and finance. Obtaining analytical solutions for SPDEs is either difficult in general or impossible, therefore effective numerical methods for studying the behaviour of these equations are of great interest for researchers. Analytical solution can be obtained for very few SPDEs in [2, 3]. Allen [1] constructed finite element and difference approximation of some SPDEs. Roth approximated the solution of some stochastic hyperbolic equations by finite

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difference methods [4]. Soheili et al. presented two methods for solving SPDEs based on Saul'yev method and a high order finite difference scheme in [5].

In this paper we use a finite difference method to discretize the stochastic Burger's equation in order to stability and the convergence of the approximate solution

$$v_t(x,t) = v_{xx}(x,t) + v(x,t)v_x(x,t) + bv_x(x,t)\dot{W}(t), \tag{1.1}$$

where $\dot{W}$ is a random process which is related to the Brownian motion $W(t)$ with $\dot{W}(t) = \frac{\partial W}{\partial t}$.

2. A finite difference method

Consider the stochastic Burger's equation (1.1) with initial condition $v(x,0) = f(x),\ 0 \le x \le 1$, and boundary conditions $v(0,t) = v(1,t) = 0,\ 0 \le t \le T$. In order to construct a stochastic finite difference scheme for SPDE (1.1) consider a uniform two dimensional mesh with step size Δx and Δt on x–axis and t–axis, respectively. Let stochastic variable u_k^n approximate $v(x,t)$ at point $(k\Delta x, n\Delta t)$, hence on this mesh we approximate

$$v_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \quad v_x(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n}{2\Delta x},$$

$$v_{xx}(k\Delta x, n\Delta t) \approx \frac{u_{k-1}^{n+1} - u_k^{n+1} - u_k^n + u_{k+1}^n}{\Delta x^2},$$

where approximation of v_{xx} is Saul'yev approximation. So (1.1) yields the approximation

$$\begin{aligned}(1+r)u_k^{n+1} - ru_{k-1}^{n+1} = {} & (1-r)u_k^n + ru_{k+1}^n \\ & + \Delta t u_k^n \Big(\frac{u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n}{2\Delta x}\Big) \\ & + \frac{b}{2\Delta x}(u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n) \\ & (W((n+1)\Delta t) - W(n\Delta t)).\end{aligned} \tag{2.1}$$

3. Consistency and stability analysis for Stochastic Scheme

Consider a SPDE in the form $Lv = F$, where L denotes the differential operator and F is an inhomogeneity. Also we represent stochastic finite difference scheme at the point $(k\Delta x, n\Delta t)$ by $L_k^n u_k^n = F_k^n$. For consistency, stability and convergence we will need a norm. Hence for sequence $x = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ the sup–norm is defined as $\|x\| =$

$\sqrt{\sup |x_k|^2}$ (see [4]). For a deeper discussion of consistency, stability and convergence we refer the reader to [4].

Theorem 3.1. *The stochastic finite difference scheme (2.1) is consistent in mean square.*

Proof. Assume that $\Phi(x,t)$ is a smooth function. Then we have

$$\begin{aligned}
&\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \\
&\leq 4\Big|\int_{n\Delta t}^{(n+1)\Delta t}\{\Phi_{xx}(k\Delta x,s) - \frac{1}{\Delta x^2}[\Phi((k-1)\Delta x,(n+1)\Delta t) \\
&- \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x,n\Delta t) + \Phi((k+1)\Delta x,n\Delta t)]\}\mathrm{d}s\Big|^2 \\
&+ 4\Big|\int_{n\Delta t}^{(n+1)\Delta t}\{\Phi(k\Delta x,s)\Phi_x(k\Delta x,s) \\
&- \Phi(k\Delta x,n\Delta t)\cdot\frac{1}{2\Delta x}[\Phi(k\Delta x,(n+1)\Delta t) \\
&+ \Phi((k+1)\Delta x,n\Delta t) - \Phi((k-1)\Delta x,(n+1)\Delta t) - \Phi(k\Delta x,n\Delta t)]\}\mathrm{d}s\Big|^2 \\
&+ 4b^2\int_{n\Delta t}^{(n+1)\Delta t}\Big|\{\Phi_x(k\Delta x,s) - \frac{1}{2\Delta x}[\Phi(k\Delta x,(n+1)\Delta t) \\
&+ \Phi((k+1)\Delta x,n\Delta t) - \Phi((k-1)\Delta x,(n+1)\Delta t) - \Phi(k\Delta x,n\Delta t)]\}\Big|^2\mathrm{d}s.
\end{aligned}$$

Since $\Phi(x,t)$ is a deterministic function, hence we have $E|L(\Phi)|_k^n - L_k^n\Phi|^2 \to 0$, as $k,\ n\to\infty$. □

Theorem 3.2. *The stochastic finite difference method (2.1) approximatin (1.1) is stable in mean square with respect to the* $\|\cdot\|_\infty = \sqrt{\sup|.|^2}$*-norm, with* $(n+1)\Delta t = t$, $r = \Delta t/\Delta x^2 \leq 1/n$, *where* $v(x,t)$ *satisfies the condition*

$$|v(x,t)v_x(x,t)| \leq C\left(|v(x,t)| + |v_x(x,t)|\right).$$

Proof. Applying $\mathbb{E}|\cdot|^2$ to (2.1) and using the independence of the Wiener process increments, we get

$$\begin{aligned}
&\mathbb{E}|(1+r)u_k^{n+1} - ru_{k-1}^{n+1}|^2 \leq (1-r)^2\mathbb{E}|u_k^n|^2 + r^2\mathbb{E}|u_{k+1}^n|^2 \\
&+ C^2\Delta t^2\mathbb{E}(|u_k^n| + \frac{|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|}{2\Delta x})^2 \\
&+ 2(1-r)C\Delta t\mathbb{E}(|u_k^n|\cdot(|u_k^n| + \frac{|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|}{2\Delta x})) \\
&+ 2rC\Delta t\mathbb{E}(|u_{k+1}^n|\cdot(|u_k^n| + \frac{|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|}{2\Delta x})) \\
&+ \frac{b^2\Delta t}{4\Delta x^2}\mathbb{E}|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|^2.
\end{aligned}$$

On the other hand

$$\mathbb{E}(|u_k^n| + \frac{|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|}{2\Delta x})^2 \leq (2 + \frac{16}{\Delta x^2}) \sup_{k,n} \mathbb{E}|u_k^n|^2, \quad (3.1)$$

and

$$\mathbb{E}(|u_k^n| \cdot (|u_k^n| + \frac{|u_k^{n+1} + u_{k+1}^n - u_{k-1}^{n+1} - u_k^n|}{2\Delta x})) \leq (1 + \frac{2}{\Delta x}) \sup_{k,n} \mathbb{E}|u_k^n|^2. \quad (3.2)$$

Therefore by (3.1) and (3.2) we get

$$\begin{aligned} &\mathbb{E}|(1+r)u_k^{n+1} - ru_{k-1}^{n+1}|^2 \\ &\leq \{1 + (2 + \frac{16}{\Delta x^2})C^2\Delta t^2 + 2(1 + \frac{2}{\Delta x})C\Delta t + \frac{8b^2\Delta t}{\Delta x^2}\} \sup_{k,n} \mathbb{E}|u_k^n|^2. \quad (3.3) \end{aligned}$$

By assuming $r = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{n}$ and using the inequalities $\Delta t \leq \frac{\Delta t}{\Delta x} \leq \frac{\Delta t}{\Delta x^2}$ we can conclude that there exists some λ such that

$$\left(2 + \frac{16}{\Delta x^2}\right) C^2 \Delta t^2 + 2\left(1 + \frac{2}{\Delta x}\right) C\Delta t + \frac{8b^2 \Delta t}{\Delta x^2} \leq \lambda^2 \Delta t.$$

So by (3.3) we have

$$\mathbb{E}|(1+r)u_k^{n+1} - ru_{k-1}^{n+1}|^2 \leq (1 + \lambda^2\Delta t) \sup_{k,n} \mathbb{E}|u_k^n|^2.$$

Since $\mathbb{E}|(1+r)u_k^{n+1} - ru_{k-1}^{n+1}|^2 \geq \sup_k \mathbb{E}|u_k^n|^2$, we get

$$\sup_k \mathbb{E}|u_k^{n+1}|^2 \leq (1 + \lambda^2\Delta t) \sup_{k,n} \mathbb{E}|u_k^n|^2.$$

Hence, by fixed n and $\Delta t = \frac{t}{n+1}$, $\mathbb{E}\|u^{n+1}\|_\infty^2 \leq \mathbb{E}\|u^0\|_\infty^2$. So the stochastic finite difference scheme (2.1) is stable. □

By considering the theorems proved for stability and consistency of the finite difference method (2.1), and according to the stochastic version of the Lax-Richtmyer theorem, the method is conditionally convergent for $\|\cdot\|_\infty$.

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SOME RESULTS ABOUT FRÉCHET Q-ALGEBRAS AND ALMOST MULTIPLICATIVE MAPS

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Abstract. In this paper, first we introduce the concept of Fréchet Q-algebra and (weakly) almost multiplicative map between Fréchet algebras. Then we investigate automatic continuity of (weakly) almost multiplicative maps on Fréchet algebras.

1. Introduction

In 1985, Jarosz introduced the concept of *almost multiplicative maps* between Banach algebras [3]. For the Banach algebras A and B, a linear map $T : A \to B$ is called *almost multiplicative* if there exists $\varepsilon \geq 0$ such that

$$\|Tab - TaTb\| \leq \varepsilon \|a\| \|b\|,$$

for all $a, b \in A$. Recently, Honary, Omidi and Sanatpour in [1, 2] investigated automatic continuity of almost multiplicative linear maps between certain topological algebras, called *Fréchet algebras*, defined as follows:

An algebra A over the complex field, which is equipped with a topology, is called a *topological algebra* if it satisfies the following conditions:

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(i) Every point of A is a closed set.
(ii) The operations addition, scalar multiplication and product on A, are continuous under the topology of A.

The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms, i.e., $p_k(xy) \leq p_k(x)p_k(y)$ for all $k \in \mathbb{N}$ and $x, y \in A$, such that $p_k(x) \leq p_{k+1}(x)$, whenever $k \in \mathbb{N}$ and $x \in A$. The Fréchet algebra A with the above generating sequence of seminorms (p_k) is denoted by $(A, (p_k))$. Banach algebras are important examples of Fréchet algebras.

For the Fréchet algebras $(A, (p_k))$ and $(B, (q_k))$, $\varepsilon \geq 0$ a linear map $T : A \to B$ is called *ε-multiplicative* with respect to (p_k) and (q_k) if

$$q_k(Tab - TaTb) \leq \varepsilon p_k(a)p_k(b),$$

for every $k \in \mathbb{N}$ and for all $a, b \in A$. We say that a linear map $T : A \to B$ is *almost multiplicative* if T is ε-multiplicative for some $\varepsilon \geq 0$. We now define another important class of topological algebras, called Q-algebras.

Definition 1.1. Let A be an algebra. An element $x \in A$ is called quasi-invertible if there exists $y \in A$ such that $x \diamond y = x + y - xy = 0$ and $y \diamond x = y + x - yx = 0$

The set of all quasi-invertible elements of A is denoted by q-Inv A. A topological algebra A is called a *Q-algebra* if q-Inv A is open, or equivalently, if q-Inv A has an interior point in A .

Definition 1.2. For a unital algebra A with the unit 1, the spectrum of an element $x \in A$, denoted by $\sigma_A(x)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - x$ is not invertible in A. For a non-unital algebra A, the spectrum of $x \in A$ is $\sigma_A(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \frac{x}{\lambda} \notin \text{q}-Inv\text{A}\}$. The spectral radius of an element $x \in A$ is $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$.

The *(Jacobson) radical* rad$A$ of an algebra A is the intersection of all maximal left (right) ideals in A. The algebra A is called *semisimple* if rad$A = \{0\}$. In the case that A is a Banach algebra we have $r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

It is known that for any algebra A

$$\text{rad}A = \{x \in A : r_A(xy) = 0, \text{ for every } y \in A\}.$$

If A is a commutative Fréchet algebra, then rad$A = \bigcap_{\varphi\in M(A)} \ker\varphi$, where $M(A)$ is the continuous character space of A, i.e. the space of all continuous non-zero multiplicative linear functionals on A.

Remark 1.3. It is interesting to note that Fréchet algebras are *lmc* algebras. Moreover, an *lmc* algebra A is a Q-algebra if and only if the spectral radius r_A is continuous at zero and it is uniformly continuous on A if A is also commutative. A Fréchet algebra A is a Q-algebra if and only if the spectral radius $r_A(x)$ is finite for all $x \in A$.

Lemma 1.4. *Let $(A,(p_k))$ be a Fréchet algebra, then*

$$r_A(x) = \sup_{k\in\mathbb{N}} lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}}$$

for every $x \in A$.

Theorem 1.5. *Let $(A,(p_k))$ be Fréchet algebra, then the following statements are equivalent:*

(i) *$(A,(p_k))$ is a Q-algebra.*
(ii) *There exists $k_0 \in \mathbb{N}$ such that $r_A(x) \leq p_{k_0}(x)$, for every $x \in A$.*
(iii) *$r_A(x) = lim_{n\to\infty} p_{k_0}(x^n)^{\frac{1}{n}}$, for every $x \in A$ and p_{k_0} as in (ii).*

2. Main results

In this section, we bring some results about Fréchet Q-algebras. Also we assume that p_{k_0} be as given in Theorem 1.5.

Lemma 2.1. *Let $(A,(p_k))$ be a Fréchet algebra, $x \in A$ such that $r_A(x) < 1$. Then for every $k \in \mathbb{N}$, $lim_{n\to\infty} p_k(x^n) = 0$.*

One can deduce the following result as a consequence of Lemma 2.1 and Theorem 1.5.

Corollary 2.2. *Let $(A,(p_k))$ be a Fréchet Q-algebra, $x \in A$ such that $p_{k_0}(x) < 1$. Then for every $k \in \mathbb{N}$, $lim_{n\to\infty} p_k(x^n) = 0$.*

Lemma 2.3. *Let $(A,(p_k))$ be a Fréchet Q-algebra and $x \in A$ such that $p_{k_0}(x) < 1$ or $r_A(x) < 1$, then $e_A - x$ is invertible.*

Corollary 2.4. *Let $(A,(p_k))$ be a Fréchet Q-algebra and $x \in A$ such that $p_{k_0}(x) \leq |\lambda|$. Then $\lambda e_A - x$ is invertible and*

$$p_{k_0}(\lambda e_A - x)^{-1} < \frac{1}{|\lambda| - p_{k_0}(x)}.$$

Definition 2.5. Let $(A,(p_k))$ be Fréchet Q-algebra with unit e_A. For $0 < \varepsilon < 1$, the $\varepsilon-$condition spectrum of an element x in A is defined by,

$$\sigma_\varepsilon(x) = \{\lambda \in \mathbb{C} : p_{k_0}(\lambda e_A - x) p_{k_0}(\lambda e_A - x)^{\frac{1}{n}} \geq \tfrac{1}{\varepsilon}\}$$

with the convention $p_{k_0}(\lambda e_A - x) p_{k_0}(\lambda e_A - x)^{\frac{1}{n}} = \infty$ when $\lambda e_A - x$ is not invertible. The $\varepsilon-$spectral radius $r_\varepsilon(x)$ is define as

$$r_\varepsilon(x) = \sup\{|\lambda| : \lambda \in \sigma_\varepsilon(x)\}.$$

Note that $\sigma_A(x) \subset \sigma_\varepsilon(x)$ and therefore $r_A(x) \leq r_\varepsilon(x)$.

Proposition 2.6. *Let $(A, (p_k))$ be a Fréchet Q-algebra, $x \in A$ and $0 < \varepsilon < 1$, then $r_\varepsilon(x) \leq (\frac{1+\varepsilon}{1-\varepsilon})p_{k_0}(x)$.*

We now present some results about (weakly) almost multiplicative maps between Fréchet algebras.

Lemma 2.7. *Let $(A, (p_k))$ be a semisimple commutative Fréchet algebra and x_0 be a none zero element in A. Then for every $\lambda \in \mathbb{C}$, there exists a weakly almost multiplicative linear functional $T : (A, (p_k)) \to \mathbb{C}$ such that $T(x_0) = \lambda$.*

Theorem 2.8. *Let $(A, (p_n))$ be a Fréchet Q-algebra and $(B, (q_n))$ be a semisimple Fréchet algebra (not necessarily commutative). Let the surjective map $T : A \to B$ be weakly almost multiplicative such that $r_B(Tx) \leq r_A(x)$ for every $x \in A$. Then T is automatically continuous.*

In the case that both algebras A and B are Fréchet Q-algebras, we have the following result.

Theorem 2.9. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet Q-algebras such that B is commutative and semisimple. If the linear map $T : A \to B$ satisfies $r_B(Tx) \leq r_A(x)$ for all $x \in A$, then T is continuous.*

Finally, we recall the following result, which concerns an interesting property of almost multiplicative linear functionals on Fréchet algebras.

Theorem 2.10. *Let $T : (A, (p_n)) \to \mathbb{C}$ be a weakly almost multiplicative linear functional. Then, at least one of the following holds:*

(i) *T is multiplicative.*
(ii) *T is continuous.*

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METRIC CHARACTERIZATIONS OF STRONG WELL-POSEDNESS FOR MULTIVALUED VARIATIONAL INEQUALITIES

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Abstract. In the present paper, we are devoted to exploring conditions of well-posedness for multivalued variational inequalities in Banach spaces. We establish some conditions under which the strong well-posedness for the considered multivalued variational inequality is equivalent to the existence and uniqueness of its solution.

1. Introduction

The classical concept of well-posedness for a global minimization problem,which was firstly introduced by Tikhonov [2] and thus has been known as the Tikhonov well-posedness, requires the existence and uniqueness of solution to the global minimization problem and the convergence of every minimizing sequence toward the unique solution. Very recently, Xiao et al. [3] established two kinds of conditions under which the strong and weak well-posedness for the hemivariational inequality are equivalent to the existence and uniqueness of its solutions, respectively.

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Let X be a Banach space and X^* its topological dual space. The norm in X and X^* will be denoted by $||\cdot||$. We denote $\langle .,. \rangle$, $[x,y]$ and $]x,y[$ the dual pair between X and X^*, the line segment for $x,y \in X$, and the interior of $[x,y]$, respectively. Now, we recall some concepts of subdifferentials that we need in the next section.

Definition 1.1 ([1])**.** Let X be a normed vector space, Ω be a nonempty subset of X, $x \in \Omega$ and $\varepsilon \geq 0$. The set of ε-normals to Ω at x is

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{ x^* \in X^* | \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x \rangle}{||u-x||} \leq \varepsilon \right\}.$$

Assume that $\overline{x} \in \Omega$, the limiting normal cone to Ω at $\overline{x}$ is

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

Let $J : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$; the limiting subdifferential of J at $\bar{x}$ is defined as follows

$$\partial_M J(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, J(\bar{x})); \text{epi} J)\}.$$

Definition 1.2. Let $T : X \to 2^{X^*}$ be a set-valued mapping. Then T is said to be relaxed monotone if there exists a constant α such that for any $x, y \in X$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v-u, x-y \rangle \leq -\alpha ||x-y||^2.$$

Definition 1.3 ([4])**.** A mapping $T : X \to X^*$ is said to be hemicontinuous if for any $x_1, x_2 \in X$, the function $t \mapsto \langle T(x_1 + tx_2), x_2 \rangle$ from $[0,1]$ into $]-\infty, +\infty[$ is continuous at 0_+.

Now, suppose that $J : X \to \mathbb{R}$, $A : X \to X^*$ and $f \in X^*$. Consider the following multivalued variational inequality associated with (A, f, J):

$MVI(A, f, J)$: Find $\bar{x} \in X$ such that for any $x \in X$, there exists $\xi \in \partial_M J(\bar{x})$,

$$\langle A\bar{x} - f + \xi, x - x \rangle \geq 0.$$

Definition 1.4. A sequence $\{x_n\} \subset X$ is said to be an approximating sequence for the $MVI(A, f, J)$, if there exists $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that for any $x \in X$ there exists $x_n^* \in \partial_M J(x_n)$,

$$\langle Ax_n - f + x_n^*, x - x_n \rangle \geq -\epsilon_n ||x - x_n||.$$

Definition 1.5. The multivalued variational inequality $MVI(A, f, J)$ is said to be strongly well-posed if it has a unique solution $\bar{x}$ on X and for every approximating sequence $\{x_n\}$, it converges strongly to x.

2. Main results

In this section, we establish some conditions under which the well-posedness for the multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

Theorem 2.1. *Let $A : X \to X^*$ be relaxed monotone with constant c and J a l.s.c. function that its limiting subdifferential satisfies relaxed monotonicity condition with constant α. If $\alpha + c > 0$, then $MVI(A, f, J)$ is strongly well-posed if and only if it has a unique solution on X.*

Example 2.2. Let $X = \mathbb{R}, f = 0$, A be the identity map and J be defined as

$$J(x) = \begin{cases} x^2 + x & \text{if } x > 0 \\ x^2 - 3x & \text{if } x \leq 0 \end{cases}$$

Then by some computation we can see that J and A are relaxed monotone with constants $\alpha = 2$, $c = 1$, respectively. Hence, all assumptions of Theorem 2.1 are fulfilled and $\bar{x} = 0$ is a unique solution of (MVI) and therefore it is strongly well-posed.

For any $\epsilon > 0$, consider the following two sets

$$\Omega(\epsilon) = \{\bar{x} | \forall x \in X, \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle A\bar{x} - f + x^*, x - \bar{x}\rangle \geq -\epsilon ||x - \bar{x}||\},$$

$$\Psi(\epsilon) = \{\bar{x} | \forall x \in X, \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle Ax - f + x^*, x - \bar{x}\rangle \geq -\epsilon ||x - \bar{x}||\}.$$

Lemma 2.3. *Suppose that $A : X \to X^*$ is monotone and hemicontinuous. Then $\Omega(\epsilon) = \Psi(\epsilon)$ for all $\epsilon > 0$.*

Lemma 2.4. *Suppose that $A : X \to X^*$ is a hemicontinuous mapping. If J is locally Lipschitz, then $\Omega(\epsilon)$ is closed in X for all $\epsilon > 0$.*

Theorem 2.5. *Suppose that $A : X \to X^*$ is hemicontinuous and monotone. Then $MVI(A, f, J)$ is strongly well-posed if and only if*

$$\forall \epsilon > 0, \, \Omega(\epsilon) \neq \emptyset, \qquad \text{and} \qquad \operatorname{diam}(\Omega(\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

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ON THE BEST PROXIMITY POINT OF SUZUKI TYPE $(\alpha, \beta, \theta, \gamma)$-CONTRACTIVE MAPPINGS

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Abstract. In this paper we investigate the existence and uniqueness of the best proximity point for Suzuki type $(\alpha, \beta, \theta, \gamma)$-contractive mappings in non-Archimedean modular metric spaces. The proved results extend and improve some well known results in the literature.

1. Introduction

Let us assume that A and B are non-empty subsets of a metric space. In view of the fact that a non-self mapping $T : A \to B$ does not necessarily have a fixed point, it is of considerable significance to explore the existence of an element x that is as close to Tx as possible. In other words, when the fixed point equation $Tx = x$ has no solution, then it is attempted to determine an approximate solution x such that the error $d(x, Tx)$ is minimum. Indeed, best proximity point theorems investigate the existence of such optimal approximate solutions, known as best proximity points, of the fixed point equation $Tx = x$, when there is no solution.

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Key words and phrases. best proximity point, modular metric space, fixed point.

Modular metric spaces are a natural and interesting generalization of classical modulars over linear spaces, like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and others. The concept of modular metric spaces was introduced in [1].

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function, for simplicity, we will write $\omega_\lambda(x, y) = \omega(\lambda, x, y)$, for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 ([1]). A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$.
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$.
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If in the above definition, we utilize the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$,

instead of (i), then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for some $\lambda > 0$.

Definition 1.2 ([1]). Suppose that ω be a pseudomodular on X and $x_0 \in X$ be fixed. Then the set

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}.$$

is called a modular space (around x_0).

Definition 1.3. Assume X_ω be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Therefore,

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$, as $n \to +\infty$ for all $\lambda > 0$. x will be called the ω-limit of (x_n).
(2) $(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to +\infty$ for all $\lambda > 0$.
(3) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.

Definition 1.4 ([2]). If in the Definition 1.1, we replace (iii) by

$$\text{(iv) } \omega_{\max\{\lambda,\mu\}}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$$

for all $\lambda, \mu > 0$ and $x, y, z \in X$, then X_ω is called a non-Archimedean modular metric space. Since (iv) implies (iii), every non-Archimedean modular metric space is a modular metric space.

We denote $A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$,
$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}$.

Definition 1.5 ([3]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Definition 1.6 ([4]). Let A and B be two nonempty subsets of a metric space (X, d) and $\alpha : A \times A \to [0, +\infty)$ be a function. We say that a non self-mapping $T : A \to B$ is triangular α-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$,

$$\text{(T1)} \begin{cases} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Longrightarrow \quad \alpha(u_1, u_2) \geq 1,$$

$$\text{(T2)} \begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \Longrightarrow \quad \alpha(x, y) \geq 1.$$

Let Θ denotes the set of all functions $\theta : R^{+^4} \to R^+$ satisfying:
(Θ_1) θ is continuous and increasing in all its variables.
(Θ_2) $\theta(t_1, t_2, t_3, t_4) = 0$ iff $t_1.t_2.t_3.t_4 = 0$.
Let $\mathcal{F}$ denotes the set of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition $\beta(t_n) \to 1$ implies $t_n \to 0$, as $n \to +\infty$.

2. Main results

Definition 2.1. Suppose that (A, B) be a pair of nonempty subsets of a modular metric space X_ω with $A_0^\lambda \neq \emptyset$ for all $\lambda > 0$. We say the pair (A, B) has the weak P_λ-property if and only if for any $x_1, x_2 \in A_0$, $y_1, y_2 \in B_0$ and $\lambda > 0$,

$$\omega_\lambda(x_1, y_1) = \omega_\lambda(A, B), \omega_\lambda(x_2, y_2) = d(A, B) \Rightarrow \omega_\lambda(x_1, x_2) \leq \omega_\lambda(y_1, y_2),$$

where $\omega_\lambda(A, B) =: \inf\{\omega_\lambda(x, y) |\ x \in A \text{ and } y \in B\}$, and $A_0^\lambda =: \{x \in A : \omega_\lambda(x, y) = \omega_\lambda(A, B) \text{ for some } y \in B\}$.

Definition 2.2. Let A and B be two nonempty subsets of a modular metric space X_ω where $A_0^\lambda \neq \emptyset$ for all $\lambda > 0$ and $\alpha : X_\omega \times X_\omega \to [0, \infty)$ is a function. A mapping $T : A \to B$ is said to be a Suzuki type $(\alpha, \beta, \theta, \gamma)-$contractive mapping if there exists $\beta \in \mathcal{F}$ and $\theta \in \Theta$ such that for all $x, y \in A$ and $\lambda > 0$ with $\frac{1}{2}\omega_\lambda^*(x, Tx) \leq \omega_\lambda(x, y)$ and $\alpha(x, y) \geq 1$ one has,

$$\omega_\lambda(Tx, Ty) \leq \beta\big(M(x, y)\big)M(x, y) + \gamma\big(N(x, y, \theta)\big)N(x, y, \theta)$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a bounded function, $\omega_\lambda^*(x,y) = \omega_\lambda(x,y) - \omega_\lambda(A,B)$,

$$M(x,y) = \max\left\{\omega_\lambda(x,y), \frac{\omega_\lambda(x,Tx)+\omega_\lambda(y,Ty)}{2} - \omega_\lambda(A,B),\right.$$
$$\left.\frac{\omega_\lambda(x,Ty)+\omega_\lambda(y,Tx)}{2} - \omega_\lambda(A,B)\right\},$$

$$N(x,y,\theta) = \theta\Big(\omega_\lambda(x,Tx) - \omega_\lambda(A,B), \omega_\lambda(y,Ty) - \omega_\lambda(A,B),$$
$$\omega_\lambda(x,Ty) - \omega_\lambda(A,B), \omega_\lambda(y,Tx) - \omega_\lambda(A,B)\Big).$$

Theorem 2.3. *Let A and B be nonempty subsets of a non-Archimedean modular metric space X_ω with ω regular such that A is $\omega-$complete and A_0^λ is nonempty for all $\lambda > 0$. Assume that T is a Suzuki type $(\alpha,\beta,\theta,\gamma)-$contractive mapping satisfying the following assertions:*

(i) *$T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$ and the pair (A,B) satisfies the weak P_λ-property,*
(ii) *T is a triangular α-proximal admissible mapping,*
(iii) *there exist elements x_0 and x_1 in A_0^λ for all $\lambda > 0$ such that,*

$$\omega_\lambda(x_1,Tx_0) = \omega_\lambda(A,B) \text{ and } \alpha(x_0,x_1) \geq 1$$

(iv) *if $\{x_n\}$ is a sequence in A s.t $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ with $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists an x^* in A such that $\omega_\lambda(x^*,Tx^*) = \omega_\lambda(A,B)$ for all $\lambda > 0$. Further, the best proximity point is unique if, for every $x,y \in A$ such that $\omega_\lambda(x,Tx) = \omega_\lambda(A,B) = \omega_\lambda(y,Ty)$, we have $\alpha(x,y) \geq 1$.

Corollary 2.4. *From Theorem 2.3, in the setting of non-Archimedean modular metric space,One can deduce Suzuki type result of Zhang et al. [3] and Suzuki type result of Suzuki [5].*

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A NONLINEAR ERGODIC THEOREM FOR A NEW CLASS OF NONLINEAR MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first consider a broad class of nonlinear mappings containing the class of (α,β)-generalized hybrid mappings in Hilbert spaces. Finally, we prove an ergodic theorem of Baillon's type for these nonlinear mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. $P_C x$ is called the metric projection of H onto C, see [3] for more details.

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* Speaker.

In this paper, motivated by Kocourek, Takahashi and Yao [2], we introduce a broad class of mappings $T : C \to C$ such that for some $\alpha, \beta, \gamma, m_1, m_2 \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (m_1 - \alpha + \gamma)\|x - Ty\|^2 - (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2$$
$$-(m_2 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \leq 0$$

for all $x, y \in C$. We call such a mapping an (α, β, γ, m_1, m_2)-generalized hybrid mapping.

Remark 1.1. The class of (α, β, γ, m_1, m_2)-generalized hybrid mappings contain many important classes of nonlinear mappings. For example nonexpansive, nonspreading, hybrid [4] and (α,β)-generalized hybrid mappings. We can also show that an (α, β, γ, m_1, m_2)-generalized hybrid mapping with a fixed point and condition $0 < m_2 + \gamma \leq m_1 + \gamma$ is quasi-nonexpansive [2].

2. Main results

In this section, using the technique developed by Takahashi [5], we prove a nonlinear ergodic theorem of Baillon's type [1] for (α, β, γ, m_1, m_2)-generalized hybrid mappings in a Hilbert space.

Theorem 2.1. *Let H be a Hilbert space and C be a closed convex subset of H. Let $T : C \to C$ be a (α, β, γ, m_1, m_2)-generalized hybrid mapping with condition $0 < m_2 + \gamma \leq m_1 + \gamma$ and $F(T) \neq \emptyset$. Let P be the metric projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since T is an (α, β, γ, m_1, m_2)-generalized hybrid mapping with condition $0 < m_2 + \gamma \leq m_1 + \gamma$ and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. So, we have that $F(T)$ is closed and convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\|PT^n x - T^n x\| \leq \|PT^{n-1}x - T^n x\| \leq \|PT^{n-1}x - T^{n-1}x\|.$$

This implies that $\|PT^n x - T^n x\|$ is a nonincreasing sequence in $\mathbb{R}$. We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u\rangle \geq 0$$

and hence

$$\|v - Pv\|^2 \leq \langle v - Pv, Pv - u\rangle.$$

So, we get

$$\begin{aligned}
\|Pv-u\|^2 &= \|Pv-v+v-u\|^2 \\
&= \|Pv-v\|^2 - 2\langle Pv-v, u-v\rangle + \|v-u\|^2 \\
&\le \|v-u\|^2 - \|Pv-v\|^2.
\end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \ge n$. Putting $v = T^m x$ and $u = PT^n x$, we have (by T is quasi-nonexpansive)

$$\begin{aligned}
\|PT^m x - PT^n x\|^2 &\le \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\
&\le \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2.
\end{aligned}$$

Since $\{\|T^n x - PT^n x\|\}$ is a nonincreasing sequence in $\mathbb{R}$, it follows that $\{PT^n x\}$ is a Cauchy sequence in $F(T)$. Therefore, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$, then we obtain for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \le \frac{1}{n}\sum_{k=0}^{n-1} \|T^k x - u\| \le \|x-u\| \ .$$

So, $\{S_n x\}$ is bounded and hence there exists a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly. If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, since $(m_2 - m_1) \le 0$, for any $y \in C$ and $k \in \mathbb{N}\bigcup\{0\}$, we have

$$\begin{aligned}
0 \le &(\beta + (\beta-\alpha)\gamma)\|T^{k+1}x - y\|^2 + (m_2 - \beta - (\beta-\alpha-1)\gamma)\|T^k x - y\|^2 \\
&- \alpha\|T^{k+1}x - Ty\|^2 - (m_1 - \alpha + \gamma)\|T^k x - Ty\|^2 \\
= &(\beta + (\beta-\alpha)\gamma)\{\|T^{k+1}x - Ty\|^2 + 2\langle T^{k+1}x - Ty, Ty - y\rangle \\
&+ \|Ty - y\|^2\} + (m_2 - \beta - (\beta-\alpha-1)\gamma)\{\|T^k x - Ty\|^2 \\
&+ 2\langle T^k x - Ty, Ty - y\rangle + \|Ty-y\|^2\} \\
&- \alpha\|T^{k+1}x - Ty\|^2 - (m_1 - \alpha + \gamma)\|T^k x - Ty\|^2 \\
= &(m_2 + \gamma)\|Ty - y\|^2 + 2\langle(\beta + (\beta-\alpha)\gamma)T^{k+1}x \\
&+ (m_2 - \beta - (\beta-\alpha-1)\gamma)T^k x - (m_2+\gamma)Ty, Ty - y\rangle \\
&+ ((\beta-\alpha) + (\beta-\alpha)\gamma)\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\}
\end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, ..., n-1$ and dividing this inequality by n, we have

$$
\begin{aligned}
0 \le\, & (m_2+\gamma)\|Ty-y\|^2 + 2\langle (m_2+\gamma)S_n x \\
& + \frac{(\beta+(\beta-\alpha)\gamma)}{n}(T^n x - x) - (m_2+\gamma)Ty, Ty-y\rangle \\
& + \frac{((\beta-\alpha)+(\beta-\alpha)\gamma)}{n}\{\|T^n x - Ty\|^2 - \|x-Ty\|^2\}
\end{aligned}
$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i}x \rightharpoonup v$ that

$$0 \le (m_2+\gamma)\|Ty-y\|^2 + 2\langle (m_2+\gamma)v - (m_2+\gamma)Ty, Ty-y\rangle.$$

Putting y=v, we have

$$0 \le -(m_2+\gamma)\|Tv-v\|^2.$$

Since $m_2+\gamma > 0$, we obtain Tv=v. To complete the proof, it is sufficient to show that if $S_{n_i}x \rightharpoonup v$, then v=p. From

$$\langle T^k x - PT^k x, PT^k x - u\rangle \ge 0$$

for all $u \in F(T)$ and Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, so we have

$$
\begin{aligned}
\langle u-p, T^k x - PT^k x\rangle &\le \langle PT^k x - p, T^k x - PT^k x\rangle \\
&\le \|PT^k x - p\|\|T^k x - PT^k x\| \\
&\le \|PT^k x - p\|\|x - Px\|.
\end{aligned}
$$

Adding these inequalities from $k=0$ to $k=n-1$ and dividing by n, we have

$$\langle u-p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x\rangle \le \frac{\|x-Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since $S_{n_i}x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u-p, v-p\rangle \le 0.$$

We know that $v \in F(T)$. So, putting $u=v$, we have

$$\langle v-p, v-p\rangle \le 0$$

and hence $\|v-p\|^2 \le 0$. So, we obtain $v=p$. This completes the proof.

$\square$

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GRAM OPERATORS OF CONTINUOUS FRAMES IN HILBERT SPACES

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Abstract. In this note, we review and study continuous frames. In particular, for two given Bessel mappings E and F, we define the so called cross-Gram operator $G_{E,F}$. We show that if F and G are Bessel mappings then $G_{E,F}$ is bounded. Also, we show that $G_{E,F}$ is bounded and invertible if and only if both E and F are of Riesz type.

1. Introduction

One of the frame related matrices or operators is the Gram matrix or operator. Unlike other related operators (analysis, synthesis and frame operator which their domain, range or both of them are in underlying Hilbert spaces), domain and range's of this operator lies in the representation sequence space ℓ^2 (in discrete case) or the function space L^2 (in continuous case), which makes it useful and sometimes hard to work.

Let us recall some definitions.

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* Speaker.

Definition 1.1. Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. The mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if

(1) F is weakly-measurable, i. e., for each $f \in \mathcal{H}$, $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω.
(2) There exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2 \qquad f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$ and Parseval if $A = B = 1$. The mapping F is called Bessel if the second inequality in (1.1) holds. In this case B is called the Bessel constant.

Theorem 1.2. *Let (Ω, μ) be a measure space and let F be a Bessel mapping from Ω to $\mathcal{H}$ with Bessel bound B. Then the operator $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by*

$$\langle T_F\phi, h\rangle = \int_\Omega \phi(\omega)\langle F(\omega), h\rangle d\mu(\omega) \qquad (h \in \mathcal{H}, \quad \phi \in L^2(\Omega, \mu))$$

is well-defined, linear, bounded with bound $\sqrt{B}$, and its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^*h)(\omega) = \langle h, F(\omega)\rangle \qquad (\omega \in \Omega, \quad h \in \mathcal{H}).$$

For a given Bessel mapping F, the operator T_F is called the pre-frame operator or synthesis operator and T_F^* is called then analysis operator of F. The operator $S_F = T_F T_F^*$ is called continuous frame operator. In case F is a continuous frame, this operator is a bounded, self-adjoint, positive, invertible operator and any $f \in \mathcal{H}$ has the resolution

$$f = \int_\Omega \langle f, F(\omega)\rangle S_F^{-1} F(\omega) d\mu = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle F(\omega) d\mu.$$

2. Gram and cross-Gram operators of continuous frames

For a given Bessel mapping $F : \Omega \to \mathcal{H}$, we can compose the synthesis operator T_F and its adjoint T_F^*, and obtain the operator

$$T_F^* T_F : L^2(\Omega, \mu) \to L^2(\Omega, \mu). \tag{2.1}$$

We call the operator $G = T_F^* T_F$ the Gram operator associated to F. Therefore for each $g, k \in L^2(\Omega, \mu)$, we have

$$\begin{aligned}
\langle T_F^* T_F(g), k\rangle &= \langle T_F(g), T_F(k)\rangle = \int_\Omega g(\omega)\langle F(\omega), T_F(k)\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\langle T_F^*(F(\omega)), k\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\langle\langle F(\omega), F(.)\rangle, k(.)\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\int_\Omega \langle F(\omega), F(\eta)\rangle \overline{k(\eta)} d\mu(\omega) d\mu(\eta) \\
&= \int_\Omega \int_\Omega g(\omega)\overline{k(\eta)}\langle F(\omega), F(\eta)\rangle d\mu(\omega) d\mu(\eta).
\end{aligned}$$

Considering the function $K : \Omega \times \Omega \to \mathbb{C}$ defined by

$$K(\omega, \eta) = \langle F(\omega), F(\eta)\rangle \quad \omega, \eta \in \Omega,$$

we can state the Gram operator (weakly) in term of the kernel function K as follows

$$\langle Gg, k\rangle = \int_\Omega \int_\Omega g(\omega)\overline{k(\eta)} K(\omega, \eta) d\mu(\omega) d\mu(\eta) \qquad (g, k \in L^2(\Omega, \mu)).$$

Lemma 2.1. *For the continuous Bessel mapping $F : \Omega \to \mathcal{H}$, the Gram operator defines an injective operator from $\mathcal{R}_{T_F^*}$ into $\mathcal{R}_{T_F^*}$ and its range is dense in $\mathcal{R}_{T_F^*}$.*

Motivating the concepts dual pairs and reproducing pairs, we compose the analysis and synthesis operators of different Bessel mappings, and define the concept cross-Gram operator. In some references the cross-Grame operator called *mixed dual Gramians* and *mixed Gramians* [5].

Let E and F be weakly measurable Bessel mappings, by composing the analysis and synthesis operators of E and F, we define the so called cross-Gram operator of E and F by $G_{E,F} = T_E^* T_F$. Let $g \in \mathrm{dom}(T_F), k \in \mathrm{dom}(T_G)$,

$$\begin{aligned}
\langle T_E^* T_F(g), k\rangle &= \langle T_F(g), T_E(k)\rangle = \int_\Omega g(\omega)\langle F(\omega), T_E(k)\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\langle T_E^*(F(\omega)), k\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\langle\langle F(\omega), E(.)\rangle, k(.)\rangle d\mu(\omega) \\
&= \int_\Omega g(\omega)\int_\Omega \langle F(\omega), E(\eta)\rangle \overline{k(\eta)} d\mu(\omega) d\mu(\eta) \\
&= \int_\Omega \int_\Omega g(\omega)\overline{k(\eta)}\langle F(\omega), E(\eta)\rangle d\mu(\omega) d\mu(\eta).
\end{aligned}$$

It is clear that, for Bessel mappings E and F with Bessel bounds B_E and B_F, the operator $G_{E,F}$ is bounded with norm at most $\sqrt{B_E B_F}$. Also $G^*_{E,F} = G_{F,E}$ and it is self adjoint and positive operator provided that $E = F$. Nevertheless, it is sometimes possible $G_{E,F}$ bounded but not both of E and F are Bessel mapping.

Theorem 2.2. *Suppose E and F are continuous frames for $\mathcal{H}$ and the corresponding cross-Gram operator $G_{E,F}$ is bounded and invertible operator. Then both E and F are Riezs-type frames for $\mathcal{H}$.*

Theorem 2.3. *Let F be a continuous frame with bounds A, B and $S^{-1}F$ be its canonical dual, then the cross-Gram operator $G_{S^{-1}F,F}$ is self-adjoint, positive operator and there is a positive operator Q such that $Q^2 = G_{S^{-1}F,F}$. Moreover Q commutes with every operator which commutes with $G_{S^{-1}F,F}$. Furthermore, if F is Riezs-type frame for $\mathcal{H}$, then $G_{S^{-1}F,F}$ is invertible.*

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COEFFICIENT BOUNDS FOR SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we intoduce two new subclasses of bi-univalent functions defined in the open unit disk. For functions belonging to theses classes we obtain estimates on first two Taylor-Maclaurian coefficients.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$.

Obviously, every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z\,(z \in \Delta)$ and $f(f^{-1}(\omega)) = \omega \;\; (|\omega| < r_0(f), r_0(f) \geq 1/4)$.

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(\omega) = w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \qquad (\omega \in \Delta)$$

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from it we conclude f^{-1} is analytic.

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if f and f^{-1} are univalent in Δ, and let Σ denote the class of bi-univalent functions in Δ of the form(1.1). For a brief history and interesting examples in the class Σ, see [3].

Some examples of functions in the class Σ are presented below

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\frac{1+z}{1-z},$$

and so on. However, the familiar Koebe function is not a member of the classΣ. Other common examples of functions in $\mathcal{S}$ such as

$$z-\frac{z^2}{2}, \qquad \frac{z}{1-z^2},$$

are also not a member of the class Σ, see [4].

In this paper we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions of subclasses of bi-univalent function.

In order to derive our main results, we have to recall here the following lemma [1,2].

Lemma 1.1. *Let $p \in \mathcal{P}$ be the family of all functions p analytic in Δ for which $\operatorname{Re}\{p(z)\} > 0$ and have the form $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$ for $z \in \Delta$. Then $|p_n| \leqslant 2$ for each n.*

2. Main results

Definition 2.1. A function $f(z)$ given by 1.1 is said to be in the class $T_\Sigma(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \left|\arg\sqrt{f'(z)}\right| < \frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left|\arg\sqrt{g'(\omega)}\right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

where $\omega = f(z)$, $g = f^{-1}$, $0 < \alpha < 1$, $\omega \in \Delta$, and the function g is given by

$$g(\omega) = w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \qquad (\omega \in \Delta)$$

Theorem 2.2. *Let $f(z)$, given by 1.1, be in the class $T_\Sigma(\alpha)$ where $0 < \alpha < 1$. Then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{1+\alpha}}, \tag{2.3}$$

and

$$|a_3| \leqslant 4\alpha^2 + \frac{4}{3}\alpha. \tag{2.4}$$

Proof. It follows from (2.1) and (2.2) that

$$\sqrt{f'(z)} = [p(z)]^{\alpha}, \tag{2.5}$$

and

$$\sqrt{g'(\omega)} = [q(\omega)]^{\alpha}, \tag{2.6}$$

where $p(z)$ and $q(\omega)$ are members of $\mathcal{P}$ with

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + q_3\omega^3 + \cdots . \tag{2.8}$$

Now, equating the coefficients in (2.5) and (2.6), we obtain

$$a_2 = \alpha p_1 \tag{2.9}$$

$$3a_3 = 2\alpha p_2 + \alpha(2\alpha - 1)p_1^2 \tag{2.10}$$

$$-a_2 = \alpha q_1 \tag{2.11}$$

$$3(2a_2^2 - a_3) = 2\alpha q_2 + \alpha(2\alpha - 1)q_1^2. \tag{2.12}$$

From (2.9) we conclude

$$p_1 = -q_1, \qquad 2a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.13}$$

Now from 2.10, 2.12 and 2.13, we obtain

$$6a_2^2 = 2\alpha(p_2 + q_2) + \alpha(2\alpha - 1)(p_1^2 + q_1^2) \tag{2.14}$$

$$= 2\alpha(p_2 + q_2) + \alpha(2\alpha - 1)\frac{2a_2^2}{\alpha^2} \tag{2.15}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2}{1+\alpha}(p_2 + q_2).$$

Applying Lemma 1.1 to the coefficients p_2 and q_2, we immediately get

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{1+\alpha}}.$$

This gives the bound on $|a_2|$ as asserted in 2.3.

Next, in order to find a bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$6a_3 - 6a_2^2 = 2\alpha(p_2 - q_2) + \alpha(2\alpha - 1)(p_1^2 - q_1^2).$$

Upon substituting the value of a_2^2 from 2.13 and observing $p_1^2 = q_1^2$, it follows that

$$6a_3 = 6a_2^2 + 2\alpha(p_2 - q_2).$$

Applying lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leqslant 4\alpha^2 + \frac{4}{3}\alpha.$$

This completes the proof of Theorem 2.2. □

Definition 2.3. A function $f(z)$ given by 1.1 is said to be in the class $T_\Sigma(\beta)$ if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \operatorname{Re}\left\{\sqrt{f'(z)}\right\} > \beta, \tag{2.16}$$

and

$$\operatorname{Re}\left\{\sqrt{g'(\omega)}\right\} > \beta, \tag{2.17}$$

where $w = f(z)$, $g = f^{-1}$, $0 \leq \beta < 1$, $w \in \Delta$.

Theorem 2.4. *Let $f(z)$, given by 1.1, be in the class $T_\Sigma(\beta), 0 \leq \beta < 1$. Then*

$$|a_2| \leqslant \sqrt{\frac{8}{6}(1-\beta)(3-2\beta)} \tag{2.18}$$

and

$$|a_3| \leqslant \sqrt{2(2-3\beta+\beta^2)^2 + \frac{4}{3}(1-3\beta+\beta^2)}. \tag{2.19}$$

Proof. With a similar argument as in the previous result, and applying Lemm1.1 for the coefficients p_1, p_2, q_1 and q_2, we can prove the theorem. □

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W-CONVERGENCE TO A MINIMIZER OF A CONVEX FUNCTION IN HADAMARD SPACES

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Abstract. w-convergence of the proximal point algorithm to a minimizer of a proper, convex and lower semicontinuous function is established in Hadamard spaces.

1. Introduction

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \longrightarrow X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic space (X, d) is a CAT(0) space if satisfies the following inequality:

CN-inequality: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

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Key words and phrases. Hadamard space, proximal point algorithm, w-convergence, subdifferential.

A complete CAT(0) space is called a Hadamard space. It is known that a CAT(0) space is an uniquely geodesic space.

For all x and y belongs to a CAT(0) space X, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$. Set $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$, a subset C of X is called convex if $[x,y] \subseteq C$ for all $x, y \in C$. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4].

A notion of convergence in Hadamrad spaces is called Δ-convergence which is characterized by Ahmadi Kakavandi [1] as follows.

Lemma 1.1 ([1])**.** *A bounded sequence (x_n) in Hadamard space (X,d), Δ-converges to $x \in X$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Berg and Nikolaev [3] introduced the concept of *quasilinearization* for CAT(0) space X. They denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle \cdot \rangle : (X\times X)\times(X\times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

We can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a,b,c \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle \leq d(a,b)d(c,d) \qquad (a,b,c,d \in X)$$

It is known ([3], Corollary 3) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space of a complete CAT(0) space X, based on a work of Berg and Nikolaev [3], as follows.

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax}\rangle \qquad (t \in \mathbb{R},\ a,b,x \in X),$$

where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)\,(t \in \mathbb{R},\ a,b \in X)$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d))$$

for all $t, s \in \mathbb{R},\ a, b, c, d \in X$. It is obtained ([2], Lemma 2.1) that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$, for all $x, y \in X$. Then, D can impose an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]) := D((t,a,b),(s,c,d))$, which is called the dual space of (X, d). Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle \quad (x^* = [t\overrightarrow{ab}] \in X, x, y \in X).$$

For all $\alpha, \beta \in \mathbb{R},\ x^*, y^* \in X^*,\ x, y \in X$, we can define

$$\langle \alpha x^* + \beta y^*, \overrightarrow{yx}\rangle := \alpha\langle x^*, \overrightarrow{yx}\rangle + \beta\langle y^*, \overrightarrow{yx}\rangle.$$

Introducing a dual for a CAT(0) space implies a concept of weak convergence with respect to the dual space, which is named w-convergence in [2]. Authors in the same paper also showed that w-convergence is stronger than Δ-convergence. Ahmadi Kakavandi in [1] represented an equivalent definition of w-convergence in complete CAT(0) spaces without using of dual space, as follows.

Definition 1.2 ([1])**.** A sequence (x_n) in the complete CAT(0) space (X, d) w-converges to $x \in X$ iff $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$ for all $y \in X$.

w-convergence is equivalent to the weak convergence in Hilbert space H, because if $(.,.)$ is the inner product in Hilbert space H, then

$$2\langle \overrightarrow{xz}, \overrightarrow{xy}\rangle = d^2(x,y) + d^2(z,x) - d^2(z,y) = 2(x - z, x - y).$$

It is obvious that convergence in the metric implies w-convergence, and in [2] it has been shown that w-convergence implies Δ-convergence but the converse is not valid (see [1]). It is known that every bounded sequence in Hadamard space X has a subsequence that is Δ-convergent. This fact is not true for w-convergence, (see [1, Example 4.7]). We say that a Hadamard space X satisfies the condition Q if every bounded sequence in X has a subsequence that is w-convergent.

It is said that a Hadamard space (X, d) satisfies the (S) property if for any $(x, y) \in X \times X$ there exists a point $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$. Hilbert spaces and symmetric Hadamard manifolds satisfy the (S) property (see [1, Definition 2.7]). Lemma 2.8 of [1] implies that if a Hadamard space (X, d) satisfies the (S) property, then it satisfies the condition Q; because every bounded sequence in a Hadamard space (X, d) has a Δ-convergent subsequence. Also, the proper Hadamard spaces satisfy the condition Q,see [1, Propositions 4.3 and 4.4].

Let X be a Hadamard space. A function $f : X \to]-\infty, +\infty]$ is called

(i) convex if $f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in X$.
(ii) proper if $D(f) := \{x \in X | f(x) < \infty\} \neq \emptyset$.
(iii) lower semicontinuous at $x \in D(f)$ if $\liminf_{y \to x} f(y) > f(x)$.

Definition 1.3 ([2])**.** Let X be a Hadamard space with dual X^* and $f : X \to]-\infty, +\infty]$ be a proper function, then the subdifferential of f is the multi-valued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle, \ z \in X\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

Theorem 1.4 (Theorem 4.2 in [2])**.** *Let $f : X \to]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then (i) f attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$. (ii) $\langle \partial f(x) - \partial f(y), \overrightarrow{yx} \rangle \geq 0$ for $x, y \in D(f)$. (iii) For any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha \overrightarrow{xy}] \in \partial f(x)$.*

2. Main results

In this section, w-convergence of the proximal point algorithm to a minimizer of a proper, lower semicontinuous and convex function is obtained in a Hademard space. Let X be a Hadamard space with dual X^*, $f : X \to]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function and (λ_n) be a sequence of positive real numbers. By part (iii) of Theorem 1.4, we can introduce the proximal point algorithm for ∂f in Hadamard space X, as follows:

$$\begin{cases} [\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}}] \in \partial f(x_n), \\ x_0 \in X. \end{cases} \tag{2.1}$$

In the following theorem, it will be shown that the sequence generated by (2.1) is w-convergent to a point of $(\partial f)^{-1}(\mathbf{0})$.

Theorem 2.1. *Let X be a Hadamard space with dual X^*, X satisfies the condition Q and $f : X \to]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function such that $(\partial f)^{-1}(\mathbf{0}) \neq \emptyset$. Suppose (λ_n) is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then the sequence generated by the proximal point algorithm (2.1) w-converges to a point $p \in (\partial f)^{-1}(\mathbf{0})$, that is, a minimizer of f.*

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Δ-CONVERGENCE OF FORWARD-BACKWARD ALGORITHM IN HADAMARD SPACES

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Abstract. We introduce the forward-backward algorithm in Hadamrad space and establish Δ-convergence of this algorithm to a common zero of two monotone operators.

1. Introduction

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \longrightarrow X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic space (X, d) is a CAT(0) space if satisfies the following inequality: CN-inequality: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

2010 *Mathematics Subject Classification.* Primary: 47H05; Secondary: 47J25, 47H09.

Key words and phrases. Hadamrad space, forward-backward algorithm, Δ-convergence, monotone operator, cocoercive operator.

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For all x and y belongs to a CAT(0) space X, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$. Set $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$, a subset C of X is called convex if $[x,y] \subseteq C$ for all $x,y \in C$. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4].

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Berg and Nikolaev [3] introduced the concept of *quasilinearization* for CAT(0) space X. They denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

We can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a,b,c \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd}\rangle \leq d(a,b)d(c,d) \qquad (a,b,c,d \in X)$$

It is known ([3], Corollary 3) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space of a complete CAT(0) space X, based on a work of Berg and Nikolaev [3], as follows.

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

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where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$ $(t \in \mathbb{R},\ a,b \in X)$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d))$$

for all $t, s \in \mathbb{R},\ a, b, c, d \in X$. It is obtained ([2], Lemma 2.1) that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle\overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle\overrightarrow{cd}, \overrightarrow{xy}\rangle$, for all $x, y \in X$. Then, D can impose an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]) := D((t,a,b),(s,c,d))$, which is called the dual space of (X,d). Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle\overrightarrow{ab}, \overrightarrow{xy}\rangle \qquad (x^* = [t\overrightarrow{ab}] \in X, x, y \in X).$$

For all $\alpha, \beta \in \mathbb{R},\ x^*, y^* \in X^*,\ x, y \in X$, we can define

$$\langle \alpha x^* + \beta y^*, \overrightarrow{yx}\rangle := \alpha\langle x^*, \overrightarrow{yx}\rangle + \beta\langle y^*, \overrightarrow{yx}\rangle.$$

Definition 1.2. Let X be a Hadamard space with dual space X^*. The multi-valued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{x \in X : A(x) \neq \emptyset\}$ is monotone iff

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0,$$

for all $x, y \in \mathbb{D}(A),\ x^* \in Ax, y^* \in A(y)$.

Definition 1.3 ([5])**.** Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator and $\lambda > 0$. The resolvent operator of A of order λ is the multi-valued mapping $J_\lambda^A : X \to 2^X$ defined by $J_\lambda^A(x) := \{z \in X : [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$.

Theorem 1.4 (Theorem 3.9 in [5])**.** *Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ is an operator. Then (1) $\mathbb{R}(J_\lambda^A) \subset \mathbb{D}(A)$ and $A^{-1}(0) = F(J_\lambda^A) := \{x \in X : J_\lambda^A x = x\}$ for every $\lambda > 0$. (2) If A is a monotone operator then J_λ^A is a single-valued and firmly nonexpansive mapping. (3) If A is a monotone operator and $0 < \lambda \leq \mu$, then*

$$d(x, J_\lambda^A(x)) \leq 2d(x, J_\mu^A(x)),\ \textit{for all } x \in X.$$

2. Main results

In the following, we introduce the forward-backward algorithm in Hadamrad spaces.

Definition 2.1. Let X be a Hadamard space with dual X^* and $A : X \to 2^{X^*}$ be a multi-valued operator and $\lambda > 0$. The operator $I - \lambda A$ is the multi-valued mapping $I - \lambda A : X \to 2^X$ defined by $(I - \lambda A)(x) := \{z \in X : [\frac{1}{\lambda}\overrightarrow{zx}] \in Ax\}$.

Let X be a Hadamard space with dual X^* and A and B are multi-valued operators such that $D(J_\lambda^B) = X$ and $D(I - \lambda A) = X$ for all $\lambda > 0$. The forward-backward algorithm for the operators A and B is generated by

$$\begin{cases} x_{n+1} = J_{\lambda_n}^B (I - \lambda_n A)(x_n), \\ x_1 \in X, \qquad (\lambda_n) \subset (0, \infty). \end{cases} \tag{2.1}$$

Definition 2.2. Let X be a Hadamard space with dual X^*. The operator $A : X \to X^*$ is β-cocoercive for $\beta > 0$ if

$$\langle Ax - Ay, yx \rangle \geq \beta D^2(Ax, Ay), \qquad (x, y \in X).$$

In the following theorem, Δ-convergence of the sequence generated by (2.1) is established.

Theorem 2.3. *Let X be a Hadamard space with dual X^* and the monotone operator B and β-cocoercive operator A are monotone operators that $A^{-1}(0) \bigcap B^{-1}(0) \neq \emptyset$, $D(J_\lambda^B) = X$ and $D(I - \lambda A) = X$ for all $\lambda > 0$. If (λ_n) is a sequence of positive real numbers such that $0 < \lambda < \lambda_n$ for all $n \in \mathbb{N}$ then the forward-backward algorithm generated by (2.1) is Δ-convergence to the unique common zero of the operators A and B.*

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WEIGHTED FRAMES IN RELATION TO CONTROLLED FRAMES IN HILBERT C^*-MODULES

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Abstract. In this paper, we investigate weighted frames and controlled frames in Hilbert C^*-modules. Using multiplier operators, we obtain some relations between weighted frames and controlled frames. Also we show that a frame weighted by a semi-normalized sequence is always a frame in Hilbert C^*-module.

1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [3] for study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer[2], and popularized from then on.

Hilbert C^*-module forms a wide category between Hilbert space and Banach space. Its structure was first used by Kaplansky in 1952. It is

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* Speaker.

an often used tool in operator theory and in operator algebra theory. It serves as a major class of examples in operator C^*-module theory.

The notions of frames in Hilbert C^*-modules were introduced and investigated by Frank and Larson [4]. They defined the standard frames in Hilbert C^*-modules in 1999 and got a series of results for standard frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. Extending the results to this more general framework is not a routine generalization, as there are essential differences between Hilbert C^*-modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^*-module. Also there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^*-modules.

Weighted frames and controlled frames in Hilbert spaces have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [1].

In this paper, we investigate weighted frames and controlled frames in Hilbert C^*-modules. We show that they share many useful properties with their corresponding notions in Hilbert spaces. We obtain a relation between weighted frames and controlled frames by using of multiplier operators. Also we show a frame weighted by a semi-normalized sequence is always a frame in Hilbert C^*-module.

2. Main results

In this section, we study the concept of weighted frames, controlled frames and multipliers of frames for controlled frames in Hilbert C^*-module. Then we investigate the relation between weighted frames and controlled frames in Hilbert C^*-modules by using of multiplier operators.

The following definition is a generalization of weighted frames in Hilbert space to Hilbert C^*-module [5].

Definition 2.1. Let $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be a sequence of elements in Hilbert C^*-module $\mathcal{H}$ on C^*-algebra A and $\{\omega_j\}_{j\in J} \subseteq Z(A)$ a sequence of positive weights, where $Z(A)$ is the center of A. This pair is called a w-frame of Hilbert C^*-module $\mathcal{H}$ if there exist constants $C, D > 0$ such that

$$C\langle f, f\rangle \leq \sum_{j\in J} \omega_j \langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle,$$

for all $f \in \mathcal{H}$.

A sequence$\{c_j : j \in J\} \in Z(A)$ is called semi-normalized if there are bounds $b \geq a > 0$ such that $a \leq |c_n| \leq b$.

Definition 2.2. Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame in Hilbert C^*-module $\mathcal{H}$ is a family of vectors $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$, such that there exist two constants $m > 0$ and $M < \infty$ satisfying

$$m\langle f, f\rangle \leq \sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq M\langle f, f\rangle,$$

for all $f \in \mathcal{H}$.

Likewise, $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ is called a C-controlled Bessel sequence with bound M if there exists $M < \infty$ such that

$$\sum_{j\in J}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq M\langle f, f\rangle,$$

for every $f \in \mathcal{H}$, where the sum in the inequality is convergent in norm.

If $m = M$, we call this C-controlled frame a tight C-controlled frame, and if $m = M = 1$ it is called a Parseval C-controlled frame.

Every frame is a I-controlled frame. Hence controlled frames are generalizations of frames.

We define controlled multipliers as follows.

Definition 2.3. Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$. Assume $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ and $\Phi = \{\phi_j \in \mathcal{H} : j \in J\}$ are C-controlled Bessel sequences for $\mathcal{H}$. Then the operator

$$M_{m,\Phi,\Psi} : \mathcal{H} \to \mathcal{H}$$

defined by

$$M_{m,\Phi,\Psi} f = \sum_{j\in J} m_j\langle f, \psi_j\rangle C\phi_j$$

is called the C-controlled multiplier operator with symbol m.

The following proposition gives a relation between controlled frames, weighted frames and multiplier operators.

Proposition 2.4. *Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$ be self-adjoint and diagonal on $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ and assume it generates a controlled frame. Then the sequence $W = \{\omega_j\}_{j\in J} \subseteq Z(A)$, which verifies the relations $C\psi_n = \omega_n\psi_n$, is semi-normalized and positive. Furthermore $C = M_{W,\tilde{\Psi},\Psi}$.*

As an application to the first part of Proposition 2.4, a frame weighted by semi-normalized sequence is always a frame. Indeed, we have the following proposition.

Proposition 2.5. *Let $\{\omega_j : j \in J\}$ be a semi-normalized sequence with bounds a and b. If $\{\psi_j : j \in J\}$ is a frame with bounds C and D in Hilbert $C*$-module $\mathcal{H}$, then $\{\omega_j\psi_j : j \in J\}$ is also a frame with bounds a^2C and b^2D. The sequence $\{\omega_j^{-1}\tilde{\psi}_j : j \in J\}$ is a dual frame of $\{\omega_j\psi_j : j \in J\}$.*

The following results give a connection between weighted frames and frame multipliers.

Theorem 2.6. *Let $\Psi = \{\psi_j \in \mathcal{H} : j \in J\}$ be a frame for Hilbert C^*-module $\mathcal{H}$. Let $m = \{m_j\}_{j\in J}$ be a positive and semi-normalized sequence. Then the multiplier $M_{m,\Psi}$ is the frame operator of the frame $\{\sqrt{m_j}\psi_j : j \in J\}$ and therefore it is positive, self-adjoint and invertible. If $\{m_j\}_{j\in J}$ is negative and semi-normalized, then $M_{m,\Psi}$ is negative, self-adjoint and invertible.*

Theorem 2.7. *Let $\{\psi_j : j \in J\}$ be a sequence of elements in Hilbert C^*-module $\mathcal{H}$. Let $W = \{\omega_j : j \in J\}$ be a sequence of positive and semi-normalized weights. Then the following properties are equivalent:*

(1) *$\{\psi_j : j \in J\}$ is a frame.*
(2) *$M_{w,\Psi}$ is a positive and invertible operator.*
(3) *There are constants $C, D > 0$ such that for all $f \in \mathcal{H}$*

$$C\langle f, f\rangle \leq \sum_{j\in J} \omega_j \langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle.$$

i.e. the pair $\{\omega_j : j \in J\}, \{\psi_j : j \in J\}$ forms a weighted frame.

(4) *$\{\sqrt{\omega_j}\psi_j : j \in J\}$ is a frame.*
(5) *$M_{w,\Psi}$ is a positive and invertible operator for any positive, semi-normalized sequence $W' = \{\omega_j'\}_{j\in J}$.*
(6) *$(\omega_j\psi_j)$ is a frame, i.e. the pair $\{\omega_j : j \in J\}, \{\psi_j : j \in J\}$ forms a weighted frame.*

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ON THE INVERTIBILITY OF RIESZ AND MODULAR RIESZ MULTIPLIERS

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Abstract. Multipliers are operators which have important applications for signal processing and acoustics. In this paper, we investigate Riesz and modular Riesz multipliers in Hilbert C^*-modules. We show, unlike to Riesz multipliers in Hilbert spaces, Riesz multipliers in Hilbert C^*-modules may not be invertible. Then we obtain some sufficient conditions for invertibility of Riesz multipliers in this setting.

1. Introduction

Hilbert C^*-modules form a wide category between Hilbert spaces and Banach spaces.

Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space $\mathcal{H}$ which is a left A-module with an inner product map $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to A$ which satisfies the following properties:

(1) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.
(2) $\langle af, g \rangle = a \langle f, g \rangle$ for all $f, g \in \mathcal{H}$ and $a \in A$.
(3) $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in \mathcal{H}$.

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(4) $\langle f, f\rangle \geq 0$ for all $f \in \mathcal{H}$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.

For $f \in \mathcal{H}$, we define a norm on $\mathcal{H}$ by $\|f\|_{\mathcal{H}} = \|\langle f, f\rangle\|_A^{1/2}$. If $\mathcal{H}$ is complete with this norm, it is called a (left) Hilbert C^*-module over A or a (left) Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^* = a$ and its spectrum is a subset of positive real numbers. In this case, we write $a \geq 0$. By condition (4) in the definition above, $\langle f, f\rangle \geq 0$ for every $f \in \mathcal{H}$, hence we can define $|f| = \langle f, f\rangle^{1/2}$. We call $Z(A) = \{a \in A : ab = ba, \forall b \in A\}$, the center of A. If $a \in Z(A)$, then $a^* \in Z(A)$, and if a is an invertible element of $Z(A)$, then $a^{-1} \in Z(A)$, also if a is a positive element of $Z(A)$, then $a^{\frac{1}{2}} \in Z(A)$. Let $\mathrm{Hom}_A(M, N)$ denotes the set of all A-linear operators from M to N.

We are focusing in finitely and countably generated Hilbert C^*- modules over unital C^*-algebra A.

The notion of (standard) frames in Hilbert C^*-modules is first defined by Frank and Larson [2].

If $\mathcal{H}$ is a Hilbert C^*-module, and J a set which is finite or countable, a system $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $C, D > 0$ such that

$$C\langle f, f\rangle \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq D\langle f, f\rangle \tag{1.1}$$

for all $f \in \mathcal{H}$. The constants C and D are called the frame bounds. If $C = D$, the frame is called a tight frame and in the case $C = D = 1$ it is called Parseval frame. It is called a Bessel sequence if the second inequality in (1.1) holds.

Definition 1.1 ([2])**.** A frame $\{f_j\}_{j\in J}$ in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A is called a Riesz basis if it satisfies:

(1) $f_j \neq 0$ for any $j \in J$.
(2) if an A-linear combination $\sum_{j\in K} a_j f_j$ is equal to zero, then every summand $a_j f_j$ is equal to zero, where $\{a_j\}_{j\in K} \subseteq A$ and $K \subseteq J$.

A. Khosravi and B. Khosravi introduced modular Riesz bases in Hilbert C^*-modules ([3]), which share many properties with Riesz bases in Hilbert spaces.

Definition 1.2. Let A be a unital C^*-algebra with identity 1_A. A sequence $\{f_j\}_{j\in J}$ in Hilbert A-module $\mathcal{H}$ is called a modular Riesz basis for $\mathcal{H}$ if there exists an invertible operator $T \in B(\ell^2(A), \mathcal{H})$ such that $Te_j = f_j$ for each $j \in J$, where $\{e_j\}_{j\in J}$ is the standard orthonormal basis of $\ell^2(A)$, i.e., $e_j = (\delta_{ij}1_A)_{j\in J}$.

In Hilbert C^*-module setting, every modular Riesz basis is a Riesz basis but every Riesz basis is not a modular Riesz basis [4].

2. MAIN RESULTS

In this section, we study the concept of multiplier for Riesz bases and modular Riesz bases in Hilbert C^*-modules and we will show some of its properties.

Definition 2.1. Let A be a unital C^*-algebra, J be a finite or countable index set and $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be Hilbert C^*-modules Bessel sequences for $\mathcal{H}$. For $m = \{m_j\}_{j\in J} \in \ell^\infty(A)$ where $m_j \in Z(A)$, for each $j \in J$, the operator $M_{m,\{f_j\},\{g_j\}} : \mathcal{H} \to \mathcal{H}$ defined by

$$M_{m,\{f_j\},\{g_j\}} f := \sum_{j\in J} m_j \langle f, f_j\rangle g_j \qquad (f \in \mathcal{H})$$

is called the multiplier operator of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$. The sequence $m = \{m_j\}$ is called the symbol of $M_{m,\{f_j\},\{g_j\}}$.

The symbol of m has important role in the studying of multiplier operators. In this paper m is always a sequence $m = \{m_j\}_{j\in J} \in \ell^\infty(A)$ with $m_j \in Z(A)$, for each $j \in J$.

One of the major question in the study of multiplier is the invertibility of multiplier operator. In the sequel, we list results about the invertibility of Riesz multipliers in Hilbert spaces.

Lemma 2.2 ([1])**.** *Let $(\psi_j) \subset \mathcal{H}_1$ be a Bessel sequence with no zero elements, and $(\phi_j) \subset \mathcal{H}_2$ a Riesz sequence. Then the mapping $m \mapsto M_{m,\phi_j,\psi_j}$ is injective from $\ell^\infty(A)$ into $B(\mathcal{H}_1, \mathcal{H}_2)$.*

Proposition 2.3 ([1])**.** *Let (ψ_j) be a Riesz basis with bounds C, D and (ϕ_j) be a frame with bounds C', D'. Then*

$$\sqrt{CC'}\|m\|_\infty \leq \|M_{m,(\phi_j),(\psi_j)}\|_{Op} \leq \sqrt{DD'}\|m\|_\infty.$$

The following lemma gives sufficient and necessary conditions for invertibility of multipliers for Riesz bases.

Theorem 2.4 ([5])**.** *Let Φ be a Riesz basis for Hilbert $\mathcal{H}$. Then the following holds.*

(1) *If Ψ is a Riesz basis for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if m is semi-normalized.*
(2) *If m is semi-normalized* $(0 < \inf |m_i| \leq \sup \inf |m_i| < \infty)$*, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if Ψ is a Riesz basis for $\mathcal{H}$.*

But the above results don't need to be true in Hilbert C^*-module setting as the following example shows.

Example 2.5. Let $A = M_{2\times 2}(\mathbb{C})$ be the C^*-algebra of all 2×2 complex matrices. Let $\mathcal{H} = A$ and for any $A, B \in \mathcal{H}$ define $\langle A, B\rangle = AB^*$. Then $\mathcal{H}$ is a Hilbert A-module. Let $E_{i,j}$ be the matrix with 1 in the (i, j)'th entry and 0 elsewhere, where $1 \le i, j \le 2$. Then $\Phi = \{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$.

It is easy to see $\Psi = \{E_{2,1}, E_{1,2}\}$ is a tight frame and therefor a Bessel sequence in $\mathcal{H}$. Let $M \in \mathcal{H}$ be arbitrary then for any $m = \{m_1, m_2\} \subset \mathbb{C}$ we have

$$M_{m,\Phi,\Psi} = m_1\langle M, E_{2,1}\rangle E_{1,1} + m_2\langle M, E_{1,2}\rangle E_{2,2} = 0.$$

Thus $M_{m,\Phi,\Psi}$ is not invertible.

In the following we obtain sufficient conditions for invertibility of Riesz multipliers in Hilbert C^*-modules.

Theorem 2.6. *Let $\Phi = \{\phi_j\}_{j\in J}$ be a Riesz basis in Hilbert C^*-module $\mathcal{H}$, then the following statements are equivalent:*

(1) $\Phi = \{\phi_j\}_{j\in J}$ *is a modular Riesz basis.*
(2) $\Phi = \{\phi_j\}_{j\in J}$ *has a unique dual frame which is a modular Riesz basis.*
(3) *Synthesis operator T_Φ is invertible.*
(4) *For analysis operator U_Φ,* $\mathrm{Rang}(U) = \ell^2(A)$.
(5) *If $\sum_{j\in J} a_j\phi_j = 0$ for some sequence $\{a_j\}_{j\in J}$, then $a_j = 0$ for each $j \in J$.*

In case the equivalent conditions are satisfied, Riesz multiplier $M_{m,\Phi,\Phi}$ is invertible where symbol $m = (m_j)$ is invertible and $|m_j|$ has a lower positive bound for each $j \in J$.

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WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS FROM $\mathcal{Q}_K(p,q)$ SPACES TO CLASSICAL WEIGHTED SPACES

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Abstract. For $-2 < q < \infty$ and $0 < p < \infty$, the $\mathcal{Q}_K(p,q)$ space is the space of all analytic functions on the open unit disk $\mathbb{D}$ satisfying

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) < \infty,$$

where $g(z,a) = \log \frac{1}{|\sigma_a(z)|}$ is the Green's function on $\mathbb{D}$ and $K : [0,\infty) \to [0,\infty)$, is a right-continuous and non-decreasing function. The boundedness and compactness of the weighted differentiation composition operators from $\mathcal{Q}_K(p,q)$ spaces and $\mathcal{Q}_{K,0}(p,q)$ spaces into the classical weighted spaces and the little classical weighted spaces are characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. For $\alpha > 0$, an $f \in H(\mathbb{D})$ is said to

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belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

$\mathcal{B}^\alpha$ is a Banach space with the above norm. When $\alpha = 1$, we get the classical Bloch space.

Let $g(z,a)$ be the Green's function with logarithmic singularity at a, i.e., $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$). Assume that $0 < p < \infty$, $-2 < q < \infty$ and $K : [0,\infty) \to [0,\infty)$ is a nondecreasing countinuous function. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_K(p,q)$ space if

$$\|f\|_K := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$. For $p \geqslant 1$, under the norm $\|f\|_{\mathcal{Q}_K(p,q)} = |f(0)| + \|f\|_K$, $\mathcal{Q}_K(p,q)$ is a Banach space. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_{K,0}(p,q)$ space if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

If $K(x) = x^s, s \geqslant 0$, the space $\mathcal{Q}_K(p,q)$ equals to the space $F(p,q,s)$ (see [5]). When $p = 2$, $q = 0$, the space $\mathcal{Q}_K(p,q)$ equals to the space $\mathcal{Q}_K$. Throughout the paper we assume that

$$\int_0^1 (1-r^2)^q K(-\log r) r dr < \infty,$$

since otherwise $\mathcal{Q}_K(p,q)$ consists only of constant functions (see [4]).

For $\alpha > 0$ the classical weighted space H_α^∞ is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f(z)| < \infty.$$

If

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f(z)| = 0,$$

it is said that f belongs to the little weighted space H_α^0. It is easy to see that both H_α^∞ and H_α^0 are Banach spaces with the norm $\|.\|_{H_\alpha^\infty}$, and H_α^0 is a closed subspace of H_α^∞. If $\alpha = 0$, then H_α^∞ becomes the space of bounded analytic functions on $\mathbb{D}$, which is denoted by $H^\infty = H^\infty(\mathbb{D})$.

Let $D = D^1$ be the differentiation operator, i.e., $Df = f'$. If n is a nonnegative integer then the operator D^n is defined by $D^0 f = f$, $D^n f = f^{(n)}$, $f \in H(\mathbb{D})$.

The weighted differentiation composition operator, denoted by $D^n_{\varphi,u}$, is defined as follows:

$$(D^n_{\varphi,u}f)(z) = u(z)f^{(n)}(\varphi(z)) \qquad (f \in H(\mathbb{D})),$$

where $u \in H(\mathbb{D})$ and φ is a nonconstant holomorphic self-map of $\mathbb{D}$.

If $n = 0$, then $D^n_{\varphi,u}$ becomes the weighted composition operator uC_φ, defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)) \qquad (z \in \mathbb{D}),$$

which for $u(z) \equiv 1$, is reduced to the composition operator C_φ.

If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. When $n = 1$ and $u(z) \equiv 1$, then $D^n_{\varphi,u} = C_\varphi D$. If $n = 1$ and $\varphi(z) = z$, then $D^n_{\varphi,u} = M_u D$, i.e. the product of differentiation operator and multiplication operator M_u defined by $M_u f = uf$.

The boundedness and compactness of the weighted differentiation composition operators can be found for example in [1, 3]. Here we study the boundedness and compactness of the weighted differentiation composition operators from $\mathcal{Q}_K(p,q)$ spaces and $\mathcal{Q}_{K,0}(p,q)$ spaces to the classical weighted spaces and the little classical weighted spaces.

2. Main Results

In this section we give main results. We adopt the methods of [1, 2]. For this purpose, we need some auxiliary results.

Lemma 2.1. *Let $\alpha, p > 0$ and $q > -2$. Then $D^n_{\varphi,u}$ from $\mathcal{Q}_K(p,q)$ ($\mathcal{Q}_{K,0}(p,q)$) into H^∞_α is compact if and only if $D^n_{\varphi,u}$ from $\mathcal{Q}_K(p,q)$ ($\mathcal{Q}_{K,0}(p,q)$) into H^∞_α is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{Q}_K(p,q)$ or $\mathcal{Q}_{K,0}(p,q)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^n_{\varphi,u}f_k\|_{H^\infty_\alpha} \to 0$ as $k \to \infty$.*

By following lemma one important property of $\mathcal{Q}_K(p,q)$ spaces is the inclusion relationship with α-Bloch spaces.

Lemma 2.2 ([4])**.** *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. For $f \in \mathcal{Q}_K(p,q)$ we have $f \in \mathcal{B}^{\frac{q+2}{p}}$ and $\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \leqslant C\|f\|_{\mathcal{Q}_K(p,q)}$.*

Furthermore, $\mathcal{Q}_K(p,q) = \mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2}K(-\log r)rdr < \infty.$$

In Theorems 2.3-2.6, K is a nonnegative nondecreasing function on $[0,\infty)$ such that

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}(\log\frac{1}{1-r})^{\chi_{-1}(q)}r dr<\infty,$$

where $\chi_o(x)$ denote the characteristic function of the set o, φ is an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$, and n is a nonnegative integer.

Theorem 2.3. *Let $\alpha,p>0$, $q>-2$. Then the following statements are equivalent.*

(i) $D^n_{\varphi,u}:\mathcal{Q}_K(p,q)\to H^\infty_\alpha$ *is bounded.*
(ii) $D^n_{\varphi,u}:\mathcal{Q}_{K,0}(p,q)\to H^\infty_\alpha$ *is bounded.*
(iii) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}<\infty.$

Theorem 2.4. *Let $\alpha,p>0$, $q>-2$. Then the following statements are equivalent.*

(i) $D^n_{\varphi,u}:\mathcal{Q}_K(p,q)\to H^\infty_\alpha$ *is compact.*
(ii) $D^n_{\varphi,u}:\mathcal{Q}_{K,0}(p,q)\to H^\infty_\alpha$ *is compact.*
(iii) $D^n_{\varphi,u}:\mathcal{Q}_K(p,q)\to H^\infty_\alpha$ *is bounded and*

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}=0.$$

Theorem 2.5. *Let $\alpha,p>0$, $q>-2$. Then $D^n_{\varphi,u}:\mathcal{Q}_{K,0}(p,q)\to H^0_\alpha$ is bounded if and only if $D^n_{\varphi,u}:\mathcal{Q}_{K,0}(p,q)\to H^\infty_\alpha$ is bounded and $g\in H^0_\alpha$.*

Theorem 2.6. *Let $\alpha,p>0$, $q>-2$. Then the following statements are equivalent.*

(i) $D^n_{\varphi,u}:\mathcal{Q}_K(p,q)\to H^0_\alpha$ *is compact.*
(ii) $D^n_{\varphi,u}:\mathcal{Q}_{K,0}(p,q)\to H^0_\alpha$ *is compact.*
(iii) $\lim_{|z|\to1}\frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}=0.$

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AN INTEGRAL-TYPE OPERATOR FROM AREA NEVANLINNA SPACES TO BLOCH-TYPE SPACES

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Abstract. Let $H(\mathbb{D})$ denote the class of all analytic functions on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let n be a nonnegative integer, φ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. The boundedness and compactness of an integral-type operator

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi \qquad (f \in H(\mathbb{D}), \quad z \in \mathbb{D}),$$

from the area Nevanlinna spaces N^p_α, where $1 \leqslant p < \infty$, $\alpha > -1$, to the Bloch-type spaces B_μ and the little Bloch-type spaces $B_{\mu,0}$, where μ is normal are characterized in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $dA(z) = \frac{1}{\pi} r dr d\theta$ the normalized Lebesgue area measure on $\mathbb{D}$.

A positive continuous function μ on $[0,1)$ is called normal if there exist $\delta \in [0,1)$ and two positive numbers a and b with $0 < a < b$, such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0.$$

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$\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$ and $\lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty$.

From now on if we say that a function $\mu : \mathbb{D} \to \mathbb{C}$ is normal we will also assume that it is radial, that is, $\mu(z) = \mu(|z|)$.

Assume $\mu : \mathbb{D} \to \mathbb{C}$ is normal. It is said that a function $f \in H(\mathbb{D})$ belongs to the Bloch-type space B_μ if

$$\|f\|_{b_\mu} = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

If

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0,$$

it is said that f belongs to the little Bloch-type space $B_{\mu,0}$. The norm on B_μ is introduced as follows

$$\|f\|_{B_\mu} = |f(0)| + \|f\|_{b_\mu}.$$

Both B_μ and $B_{\mu,0}$ become Banach spaces with the norm $\|.\|_{B_\mu}$, and $B_{\mu,0}$ is a subspace of B_μ. For $\mu(z) = (1-|z|^2)^\alpha, \alpha > 0$, B_μ becoms the α-Bloch space B^α, which for $\alpha = 1$ becoms the classical Bloch space B. For $\alpha > 0$, $\beta \geqslant 0$ and

$$\mu(z) = (1-|z|)^\alpha \left(\ln \frac{e^{\frac{\beta}{\alpha}}}{1-|z|}\right)^\beta,$$

B_μ is the logarithmic Bloch-type space $B^\alpha_{\log^\beta}$.

A non-negative function $\|.\|$ on a vector space X (over the real or complex field K) is called an F-norm if the following properties are satisfied:

(i) $\|x\| = 0 \Leftrightarrow x = 0$.
(ii) $\|\lambda x\| \leqslant \|x\|$ for all $\lambda \in K$ with $|\lambda| \leqslant 1$.
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.
(iv) If $\lambda_m \to 0$ and $\lambda_m \in K$, then $\|\lambda_m x\| \to 0$.

An F-norm $\|.\|$ induces a transitive invariant distance d by $d(x, y) = \|x - y\|$ for all $x, y \in X$. A vector space X with an F-norm $\|.\|$ is said to be an F^*-space. A complete F^*-space is called an F-space.

Let $1 \leqslant p < \infty$, $\alpha > -1$. A function $f \in H(\mathbb{D})$ is said to belong to the area Nevanlinna space $N^p_\alpha(\mathbb{D}) = N^p_\alpha$, if

$$\|f\|^p_{N^p_\alpha} = \int_{\mathbb{D}} [\log(1 + |f(z)|)]^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$. It is easy to see that

$$\|f + g\|_{N^p_\alpha} \leqslant \|f\|_{N^p_\alpha} + \|g\|_{N^p_\alpha} \tag{1.1}$$

for all $f, g \in N_\alpha^p$. Consequently, N_α^p becomes a metric space. Also, we have by subharmonicity

$$\log(1+|f(z)|) \leqslant C\frac{\|f\|_{N_\alpha^p}}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \qquad (f \in N_\alpha^p), \tag{1.2}$$

where C depends only on p and α (see, e.g. [1]). From (1.1) and (1.2), it is easy to verify that $\|.\|_{N_\alpha^p}$ is an F-norm of the space N_α^p. Moreover, from (1.2), it follows that if $f_m \to f$ in N_α^p, then $f_m \to f$ locally uniformly. Here, locally uniform convergence means uniform convergence on every compact subset of $\mathbb{D}$. Therefore, under the F-norm, N_α^p becomes an F-space, i.e., a translation-invariant complete metric space.

Let $n \in \mathbb{N}_0$, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The integral-type operator is defined by

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi \qquad (f \in H(\mathbb{D}), \quad z \in \mathbb{D}).$$

When $n = 1$, $C_{\varphi,g}^n$ is the generalized composition operator C_φ^g as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi \qquad (f \in H(\mathbb{D}), \quad z \in \mathbb{D}).$$

When $g = \varphi'$, the generalized composition operator C_φ^g is the composition operator C_φ which is defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \qquad (f \in H(\mathbb{D}), \quad (z \in \mathbb{D}).$$

The boundedness and compactness of the integral-type operator $C_{\varphi,g}^n$ can be found for example in [2, 3, 4].

It is interesting to provide a function theoretic characterization of the conditions under which the integral-type operator $C_{\varphi,g}^n$ becomes a bounded or compact operator on various spaces of analytic functions. This paper focuses on the boundedness and compactness of the integral-type operator $C_{\varphi,g}^n$ from the area Nevanlinna spaces to the Bloch-type spaces and the little Bloch-type spaces.

2. Main results

In order to formulate our main results we need some auxiliary results which are incorporated in the following lemmas.

Lemma 2.1. *Suppose that $g \in H(\mathbb{D})$, μ is normal, φ is an analytic self-map of $\mathbb{D}$, and $1 \leqslant p < \infty$, $\alpha > -1$. Then the integral-type operator $C_{\varphi,g}^n : N_\alpha^p \to B_\mu$ is compact if and only if $C_{\varphi,g}^n : N_\alpha^p \to B_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in N_α^p which converges to zero locally uniformly on $\mathbb{D}$, we have $\|C_{\varphi,g}^n f_k\|_{B_\mu} \to 0$ as $k \to \infty$.*

Lemma 2.2 ([5]). *Let $n \in \mathbb{N}_0$, $1 \leqslant p < \infty$ and $\alpha > -1$. Then there exists some positive constant C independent of f such that for each $f \in N_\alpha^p$, and $z \in \mathbb{D}$,*

$$|f^{(n)}(z)| \leqslant \frac{1}{(1-|z|^2)^n} \exp\Big[\frac{C\|f\|_{N_\alpha^p}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}\Big].$$

We adopt the methods of [5] to obtain our main results.

Theorem 2.3. *Suppose that $n \in \mathbb{N}_0$, $1 \leqslant p < \infty$, $\alpha > -1$, $g \in H(\mathbb{D})$, μ is a normal function, and φ is an analytic self-map of $\mathbb{D}$. Then the integral type operator $C_{\varphi,g}^n : N_\alpha^p \to B_\mu$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \infty.$$

Theorem 2.4. *Suppose that $n \in \mathbb{N}_0$, $1 \leqslant p < \infty$, $\alpha > -1$, $g \in H(\mathbb{D})$, μ is a normal function, and φ is an analytic self-map of $\mathbb{D}$. Then the integral-type operator $C_{\varphi,g}^n : N_\alpha^p \to B_\mu$ is compact if and only if $C_{\varphi,g}^n : N_\alpha^p \to B_\mu$ is bounded and for all $C > 0$,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0.$$

Theorem 2.5. *Suppose that $n \in \mathbb{N}_0$, $1 \leqslant p < \infty$, $\alpha > -1$, $g \in H(\mathbb{D})$, μ is a normal function, and φ is an analytic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) *$C_{\varphi,g}^n : N_\alpha^p \to B_{\mu,0}$ is bounded.*
(ii) *$C_{\varphi,g}^n : N_\alpha^p \to B_{\mu,0}$ is compact.*
(iii) *$\lim_{|z|\to 1} \mu(z)|g(z)| = 0$ and for all $C > 0$,*

$$\lim_{|z|\to 1} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0.$$

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L*-LIMITED PROPERTY IN BANACH SPACES

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Abstract. Introducing the concept of L*-limited sets and L*-limited Banach spaces, we obtain some characterizations of it with respect to some well known geometric properties of Banach spaces, such as, DP* property, Gelfand-Phillips property, L-limited property and etc.

1. Introduction

A subset A of a Banach space X is called limited (resp., Dunford-Pettis (DP)), if every weak* null (resp., weak null) sequence (x_n^*) in X^* converges uniformly on A, that is,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Also if $A \subseteq X^*$ and every weak null (resp., weak null and limited) sequence (x_n) in X converges uniformly on A, we say that A is an L-set (resp., L-limited).

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* Speaker.

We know that every relatively compact subset of X is limited and clearly every limited set is DP and every DP subset of a dual Banach space is an L-set and every L-set is an L-limited set. Also every relatively weakly compact subset of a dual Banach space is an L-limited set, but the converse of these assertions, in general, are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand-Phillips (GP) property. For example, the classical Banach spaces c_0 and ℓ_1 have the GP property and every reflexive space, every Schur space (i.e., weak and norm convergence of sequences in X coincide), and dual of spaces containing no copy of ℓ_1, have the same property.

Recall that a Banach space X is said to have the DP property if every weakly compact operator $T : X \to Y$ is completely continuous (that is, T maps weakly null sequences into norm null sequences) and X is said to have the reciprocal Dunford-Pettis property (RDP) if every completely continuous operator on X is weakly compact.

So the Banach space X has the DP property if and only if every relatively weakly compact subset of X is DP and it has the RDP property if and only if every L-set in X^* is relatively weakly compact.

A stronger version of DP property was introduced by Borwein, Fabian and Vanderweff in [1]. In fact, a Banach space X has the DP* property if every relatively weakly compact subset of X is limited. But if X is a Grothendieck space (i.e., weak and weak* convergence of sequences in X^* coincide), then these properties are the same on X. The reader can find some useful and additional properties of limited and DP sets and Banach spaces with the GP, DP or RDP property in [1, 2, 3].

We recall from [5] that a Banach space X has the L-limited property if every L-limited subset of X^* is relatively weakly compact, and a bounded linear operator $T : X \to Y$ is limited completely continuous (lcc) if it carries limited and weakly null sequenses in X to norm null ones in Y[4]. We denote the class of all limited completely continuous operators from X to Y by $\mathrm{Lcc}(X, Y)$. It is clear that every completely continuous operator is lcc and the authors in [4] have shown that every weakly compact operator is limited completely continuous.

2. Main results

Definition 2.1. A subset A of a Banach space X is called an L*-limited set, if every weak null and limited sequence (x_n^*) in X^* converges uniformly on A.

It is clear that every DP set in X is L*-limited and every subset of an L*-limited set is the same. Also, it is evident that every L*-limited set is bounded. The following theorem gives some aditional properties of these concepts.

Theorem 2.2. (a) *Relatively weakly compact subsets of a Banach space are* L**-limited.*
(b) *Absolutely closed convex hull of an* L**-limited set is* L**-limited.*

Note that the converse of assertion (a) in general, is false. In fact, the following theorem shows that the closed unit ball of c_0 is an L*-limited set, but it is not relatively weakly compact.

Theorem 2.3. *A dual space X^* has the* GP *property iff every bounded subset of X is an* L**-limited set.*

Definition 2.4. Let X and Y be arbitrary Banach spaces. An operator $T : X \to Y$ is called L*-limited if $T(B_X)$ is L*-limited in Y. We denote the class of all L*-limited operators from X to Y by $L^*_{\mathrm{li}}(X, Y)$.

It is clear that an operator T is L*-limited iff T^* is lcc. Also each weakly compact operator is L*-limited.

Theorem 2.5. *A Banach space X^* has the* DP* *property iff each* L**-limited set in X is a* DP *set.*

Definition 2.6. A Banach space X has the L*-limited property, if every L*-limited set in X^* is relatively weakly compact.

Theorem 2.7. *For a Banach space X, the following are equivalent:*
(a) *X has the* L**-limited property,*
(b) *For each Banach space Y, $L^*_{\mathrm{li}}(Y, X) = W(Y, X)$,*
(c) *$L^*_{\mathrm{li}}(\ell_1, X) = W(\ell_1, X)$.*

The following corollary shows that the Banach spaces c_0 and ℓ_1 do not have the L*-limited property.

Corollary 2.8. *A Banach space X is reflexive if and only if X has the* L**-limited property and its dual has the Gelfand-Phillips property.*

Theorem 2.9. *If a Banach space X has the* L*-limited property, then it has the* L**-limited property.*

Theorem 2.10. *If X has the* L*-limited property, then its dual has the* L**-limited property.*

As a corollary, since ℓ_∞ has the L-limited property, it has L*-limited property. This shows that L*-limited property on Banach spaces is not hereditary as c_0 does not have this property.

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SOLVING VECTOR VARIATIONAL INEQUALITY AS A NECESSARY CONDITION FOR SOLVING VECTOR OPTIMIZATION PROBLEMS

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Abstract. We generalize the Stampacchia vector variational inequality (SVVI) defined by means of Clarke directional derivative. By using two new constraint qualifications, we prove that solution for SVVI is a necessary condition for solving a multiobjective problem, in both weak and strong forms.

1. Introduction and Preliminaries

Variational inequality has shown to be an important mathematical model in the study of many real problems, in particular, equilibrium problems. Several well-known problems from mathematical programming, such as system of nonlinear equations, optimization problems, complementarity problems, and fixed point problems, can be written in the form of a variational inequality problem. In the recent decades,

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* Speaker.

a variational inequality problem and its various generalizations and extensions have been studied and analyzed quite extensively, see [2, 3].

Let M be a subset of $\mathbb{R}^\ell$. As usual, $\overline{M}$ will denote the closure of M. The negative polar and the strictly negative polar of M are defined, respectively, by

$$\begin{aligned} M^- &:= \Big\{\xi \in \mathbb{R}^\ell : \forall v \in M,\ \langle \xi, v\rangle \leq 0\Big\},\\ M^s &:= \Big\{\xi \in \mathbb{R}^\ell : \forall v \in M,\ \langle \xi, v\rangle < 0\Big\},\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denoted the inner product in $\mathbb{R}^\ell$.

To facilitate our discussion, some standard notions in nonsmooth analysis are reviewed. Let $\varphi : \mathbb{R}^\ell \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of φ at x in the direction v is defined as

$$\varphi^\circ(x; v) := \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{\varphi(y+tv)-\varphi(y)}{t}.$$

The Clarke subdifferential of φ at x is defined by

$$\partial_C\varphi(x) := \{\xi \in \mathbb{R}^\ell : \forall v \in \mathbb{R}^\ell,\ \langle \xi, v\rangle \leq \varphi^\circ(x; v)\ \ \}.$$

Let $M \subset \mathbb{R}^\ell, x_0 \in \overline{M}$. The contingent cone of M at x_0 is

$$T(M, x_0) := \Big\{v \in \mathbb{R}^\ell : \exists t_n \downarrow 0, \exists v_n \longrightarrow v;\ x_0 + t_n v_n \in M\Big\}.$$

Here, we consider the following multiobjective programming problem:

$$\begin{array}{lll} \textbf{(MP)} & \min & f(x) = (f_1(x), \ldots, f_m(x)),\\ & \text{s.t.} & g(x) = (g_1(x), \ldots, g_n(x)) \leqq 0,\\ & & h(x) = (h_1(x), \ldots, h_p(x)) = 0,\\ & & x \in Q,\end{array}$$

where $f_i,\ i \in I = \{1, \ldots, m\},\ g_j,\ j \in J = \{1, \ldots, n\},\ h_k,\ k \in K = \{1, \ldots, p\}$ are locally Lipschitz real-valued functions from $\mathbb{R}^l$ and Q is an arbitrary set in $\mathbb{R}^l$. The feasible region of MP is defined as

$$S := \Big\{x \in \mathbb{R}^\ell : g(x) \leqq 0, h(x) = 0, x \in Q\Big\}.$$

Definition 1.1. The feasible point x_0 is said to be

- a local efficient solution for MP iff there exists a neighbourhood U of x_0 such that for any $x \in U \cap S$, the inequality $f(x) \leq f(x_0)$ does not hold.

- a local weak efficient (or weak efficient) solution for MP iff there exists a neighbourhood U of x_0 such that for any $x \in U \cap S$, the inequality $f(x) < f(x_0)$ does not hold.

The Stampacchia vector variational inequality in both weak and strong forms can be defined as follows:

- Stampacchia vector variational inequality (SVVI):
find some $x_0 \in S$, such that for each $x \in S$, the following inequality does not hold
$$f^{\circ}(x_0; x - x_0) \leq 0. \tag{1.1}$$
- Weak Stampacchia vector variational inequality (WSVVI):
find some $x_0 \in S$ such that for each $x \in S$, the following inequality does not hold
$$f^{\circ}(x_0; x - x_0) < 0. \tag{1.2}$$

Ansari and Lee in [1, Theorem 4.5] showed that in the absence of equality and inequality constraints, if Q is convex and $-f$ is strictly h-convex at x_0, where h is a bifunction, then the solution to the Stampacchia VVI is a necessary optimality condition for local efficiency. We define constraint qualifications and prove that, under these constraint qualifications, the solution to the SVVI (WSVVI) is a necessary optimality condition for local efficiency (local weak efficiency).

Definition 1.2. Let us introduce the following constraint qualifications:
$$\Big(\bigcup_{i \in I} \partial_C f_i(x_0)\Big)^{-} \subseteq \bigcap_{i=1}^{m} T(S^i, x_0), \tag{CQ1}$$
$$\Big(\bigcup_{i \in I} \partial_C f_i(x_0)\Big)^{s} \subseteq T(S, x_0), \tag{CQ2}$$
where
$$S^l := \Big\{x \in \mathbb{R}^{\ell} : \forall i \neq l,\ f_i(x) \leq f_i(x_0),\ g(x) \leqq 0, h(x) = 0, x \in Q\Big\}.$$

2. Main results

Now, it is time to state the main results.

Theorem 2.1. *Let x_0 be a local efficient solution for* MP*, and* (CQ1) *holds at x_0; then x_0 solves* SVVI.

We use the concept of local weak efficiency and, following the proof of Theorem 2.1, we derive similar theorem about WSVVI.

Theorem 2.2. *Let x_0 be a local weak efficient solution for* MP*, and* (CQ2) *holds at x_0; then x_0 solves* WSVVI.

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STRONG CONVERGENCE THEOREM FOR GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, a new iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in Hilbert spaces is introduced. A strong convergence theorem is given for this method. This improves and extends some recent results.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. In this paper, we assume B is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in C$. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0, \quad \text{for all } v \in C. \tag{1.1}$$

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* Speaker.

The set of solutions of (1.1) is denoted by $EP(\phi)$. The equilibrium problem (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors have proposed some useful methods for solving the equilibrium problem (1.1); see [2].

A mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in H$. Let $F(T)$ denote the fixed points set of T. Recall that a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq k\|x - y\|$, for all $x, y \in H$, where $k \in (0, 1)$ is a constant.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$, for all $y \in C$. Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y\rangle \geq 0, \quad \text{for all } y \in C.$$

Definition 1.1. Let H be a real Hilbert space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 1$, define a mapping W_n of H into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.
\end{aligned}$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$; see [4].

In 2013, Razani and Yazdi [5] introduced an iterative scheme by

$$\begin{cases} \phi(u_n, x) + \frac{1}{r_n}\langle x - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } x \in C,\\ y_n = (1 - \gamma_n)x_n + \gamma_n W_n u_n,\\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n y_n, \end{cases} \tag{1.2}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, f is a contraction of C into itself, A is a strongly positive bounded linear operator on C, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$, $\{r_n\} \subset (0, \infty)$ and W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$. They proved the sequences

$\{x_n\}$ and $\{u_n\}$ generated by (1.2), converge strongly to $x^* \in F$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \bigcap EP(\phi)}(I - A + \gamma f)(x^*)$.

2. Main results

In this paper, we prove a strong convergence theorem, concerning a new iterative scheme, for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. In order to do this, we recall some definitions as follows.

A generalized equilibrium problem is to find $z \in C$ such that

$$\phi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \text{for all } y \in C, \tag{2.1}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction and $A : C \to H$ is a monotone map. The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : \phi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \text{for all } y \in C\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(\phi)$. Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in noncooperative games and economics reduce to find a solution of (2.1) (see, for instance, [3]).

A mapping $A : C \to H$ is called α-inverse-strongly monotone, if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$

Lemma 2.1 ([1])**.** *Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(A_1) *$\phi(x, x) = 0$ for all $x \in C$.*

(A_2) *ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$.*

(A_3) *for each $x, y, z \in C$,*

$$\lim_{t \downarrow 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y).$$

(A_4) *for each $x \in C, y \mapsto \phi(x, y)$ is convex and weakly lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad \textit{for all } y \in C.$$

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$, B be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ such that $\|B\| \leq 1$, A be an α-inverse-strongly* *monotone on C and f be a contraction of C into itself with constant $k \in (0,1)$. Assume $0 < \gamma < \overline{\gamma}/k$. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \bigcap EP \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset [a,b] \subset (0, 2\alpha)$ is a real sequence satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

(iv) $0 < \liminf_{n\to\infty} r_n$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Let $x_0 \in C$. Then, the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle Ax_n, y - u_n\rangle \geq 0, \quad \textit{for all } y \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n W_n u_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n y_n, \end{cases} \tag{2.2}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$, converge strongly to $x^* \in F$, where $x^* = P_{\bigcap_{n=1}^{\infty} F(T_n)\bigcap EP}(I - B + \gamma f)(x^*)$.

Remark 2.3. Theorem 2.2 is a generalization of [5, Theorem 2.11]. To see this, set $A = 0$ in Theorem 2.2.

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ON A GENERALIZATION OF COMPACT OPERATORS

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ABSTRACT. Quasi-Cauchy sequences and ward compact sets are introduced recently by D. Burton and J. Coleman and independently by H. Cakalli. Using this concepts, we define ward compact operators between Banach spaces as a generalization of compact operators and consider some of their properties. Also, we prove some new results about ward compact oprators and give some examples of ward compact operators which are not compact.

1. INTRODUCTION

Quasi-Cauchy sequences are introduced by D. Burton in 2010 [1], as a generalization of Cauchy sequenses. After then, it has been published many papers about the subject; see [3, 4]. The concept of ward continuity of a real function and ward compactness of a subset of a set of real numbers are introduced by Cakalli in [2].

The compact linear operators are very important and have many applications. For instance, they play a central role in the theory of integral equations and in other branches of mathematical physics. Now it is natural to study some well-known properties of compact operators with respect to this new concept of compactness.

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Key words and phrases. compact operator, compact set, ward compact operator, ward compact set.

2. Definition and preliminaries

In this section, we collect some definitions and previosly proved results which we need in the next section. We suppose that X and Y are metrizable topological vector spaces with metrics d_X and d_Y.

Definition 2.1. A sequenc (x_n) is called quasi-Cauchy if for every given $\epsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies that $d(x_{n+1}, x_n) < \epsilon$.

Example 2.2. The sequence $s_n = \sum_{i=1}^{n} 1/i$ is quasi-Cauchy because $s_{n+1} - s_n = 1/(n+1)$ which tends to zero, but it is well known that it is not a Cauchy sequence.

Definition 2.3. An operator $T : X \to Y$ is said to be compact if for any bounded set B the closure of $T(B)$ is compact.

Definition 2.4. A subset E of X is called ward compact if any sequence of points in E has a quasi-Cauchy subsequenc.

It is well known that in a compact set every sequence has a Cauchy subsequence. Since every Cauchy sequence is quas-Cauchy, then every compact set is ward compact but not vice versa.

It is easily seen that any finite subset of X is ward compact, the union of two ward compact subsets of X is ward compact and the intersection of any family of ward compact subsets of X is ward compact. Furthermore, any subset of a ward compact set is ward compact.

Any compact subset of $\mathbb{R}$ is also ward compact and the converse is not always true and there are ward compact subsets of $\mathbb{R}$ which are not compact, as the following example shows.

Example 2.5. The set $K = \{\sqrt{n} : n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$ is ward compact, but it is not compact.

Let f be a real function defned on $\mathbb{R}$. This function is continuous if and only if it preserves Cauchy sequences, i.e., $(f(x_n))$ is a Cauchy sequence whenever (x_n) is. Using the idea of continuity of a real function in terms of sequences in the sense that a function preserves a certain kind of properties of sequences in the above manner, ward continuity of a function can be defined.

Definition 2.6. A function $f : X \to Y$ is called ward continuous if it preserves quasi-Cauchy sequences, i.e., $(f(x_n))$ is a quasi-Cauchy sequence whenever (x_n) is.

Definition 2.7. Let X and Y be topological vector spaces. An operator $T : X \to Y$ is said to be ward compact operator if for any bounded set B the closure of $T(B)$ is ward compact.

Since every Cauchy sequence is quasi-Cauchy, then every compact operator is ward compact but not vice versa. An example of a ward compact operator which is not compact is given in the next section.

The following theorem which is proved in [4] is our main tool to prove separability of range of a ward compact operator.

Theorem 2.8 ([4]). *If X is ward compact, then it is separable.*

3. Main results

One of the most important properties of compact operators is the separability of their range. In this section, we consider this property and some other proprties for ward compact operators. We have proved the following criterion [5] and we will use it to prove our results here.

Theorem 3.1 (Ward compactness criterion, [5]). *Let X and Y be topological vector spaces and $T : X \to Y$ a linear operator. Then T is ward compact iff it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a quasi-Cauchy subsequence.*

Now we are ready to state and prove the separability of range of a ward compact operator.

Theorem 3.2 (Separability of range). *The range $\mathcal{R}(T)$ of a ward compact linear operator $T : X \to Y$ is separable; here, X and Y are normed spaces.*

Proof. Consider the ball $B_n = B(0; n) \subset X$. Since T is ward compact, the closure of the image $C_n = T(B_n)$, that is, $\overline{T(B_n)}$ is ward compact. By [3], C_n is separable. The norm of any $x \in X$ is finite, so that $||x|| < n$, hence $x \in B_n$ for sufficiently large n. Also,

$$X = \bigcup_{n=1}^{\infty} B_n, \qquad T(X) = \bigcup_{n=1}^{\infty} T(B_n).$$

Since C_n is separable, it has a countable dense subset D_n, and the union $D = \bigcup D_n$ is countable. Since D is dense in the range $\mathcal{R}(T) = T(X)$, $\mathcal{R}(T)$ is separable. □

Lemma 3.3 (Ward compactness of product). *Let $T : X \to X$ be a ward compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS and ST are ward compact.*

Proof. Let $B \subset X$ be any bounded set in X. Since S is a bounded operator, $S(B)$ is a bounded set, and the set $T(S(B)) = TS(B)$ is ward compact because T is ward compact. Hence, TS is a ward compact linear operator. We prove that ST is also ward compact. Let (x_n) be any bounded sequence in X. Then (Tx_n) has a quasi-Cauchy subsequence (Tx_{n_k}), by the continuity of S the sequence (STx_{n_k}) is quasi-Cauchy. Hence, ST is ward compact.

□

Theorem 3.4. *Let (T_n) be a sequence of ward compact linear operators from a normed space X into a Banach space Y. Let (T_n) be uniformly operator convergent to T, that is, $\|T_n - T\| \to 0$, then the limit operator T is ward compact.*

Proof. It is obvious. □

By following examples we show that the notion of ward compact operators is different from the compact one.

Example 3.5.
(1) $T : l^2 \to l^2$ defined by $T(e_n) = \sqrt{n}e_n$ is ward compact, but it is not compact.
(2) By a famous theorem on operator theory, the operator $T(e_n) = \lambda_n e_n$ is compact if and only if $\lambda_n \to 0$, so this operator is not compact.
(3) The identity operator I on an infinite dimentional space is not ward compact.

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Posters

ON THE GEOMETRIC MEAN OF THE PRIMITIVE CHARACTERS

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Abstract. We denote by G_n the geometric mean of the numbers $\phi^*(1), \phi^*(2), \ldots, \phi^*(n)$, where ϕ^* denotes the number of primitive characters modulo q. In this paper, we prove that

$$G_n = An + O\Big(\frac{n}{\log n}\Big),$$

where A is a absolute constant.

1. Introduction and Main results

A Dirichlet character χ mod q is said to be primitive mod q if and only if for every divisor d of q, $0 < d < q$, there exists an integer $a \equiv 1 \pmod d$, $(a, q) = 1$, such that $\chi(a) \neq 1$. The number of primitive characters modulo q, $\phi^*(q)$, is given by (see [3])

$$\phi^*(q) = q \prod_{p \| q} \Big(1 - \frac{2}{p}\Big) \prod_{p^2 | q} \Big(1 - \frac{1}{p}\Big)^2. \tag{1.1}$$

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H. Jager [2] in 1973 studied the average order of the number of primitive characters and proved that

$$\sum_{n\leqslant x}\phi^*(n)=\frac{18}{\pi^4}x^2+O\Big(x\log^2 x\Big).$$

In this paper we study the geometric mean of the number of primitive characters. We prove the following average result.

Theorem 1.1. *There exists an absolute constant K such that*

$$\sum_{k\leqslant n}\log\phi^*(k)=n\log n+Kn+O\Big(\frac{n}{\log n}\Big).$$

As an immediate corollary, we obtain the following.

Corollary 1.2. *We have*

$$G_n=e^K n+O\Big(\frac{n}{\log n}\Big).$$

To prove Theorem 1.1, we need some auxiliary facts. We have

$$\sum_{p\leqslant x}\frac{\log p}{p}=\log x+E+O\Big(\frac{1}{\log x}\Big), \tag{1.2}$$

where

$$E=-\gamma-\sum_{n=2}^{\infty}\sum_{p}\frac{\log p}{p^n}.$$

For the Chebyshev's function, which is defined by $\theta(x)=\sum_{p\leqslant x}\log p$, we have

$$\theta(x)=\sum_{p\leqslant x}\log p=x+O\Big(\frac{x}{\log x}\Big). \tag{1.3}$$

We recall that for each real $\alpha>1$ and each real $z>1$ we have

$$\sum_{p>z}\frac{\log p}{p^\alpha}\ll\frac{1}{z^{\alpha-1}}. \tag{1.4}$$

Also, for $x\geqslant 2$ we have

$$\sum_{p\leqslant x}\Big\{\frac{x}{p}\Big\}=(1-\gamma)\frac{x}{\log x}+O\Big(\frac{x}{\log^2 x}\Big). \tag{1.5}$$

2. Proof of the main result

We know that $\phi^*(q)$ is multiplicative; so the function $\log\phi^*(q)$ is additive. Hence

$$\begin{aligned}
\sum_{k\leqslant n}\log\phi^*(k) &= \sum_{\substack{p^a\leqslant n\\ a\geqslant 1}}\Big(\log\phi^*(p^a)-\log\phi^*(p^{a-1})\Big)\left[\frac{n}{p^a}\right]\\
&= \sum_{p\leqslant n}\Big(\log\phi^*(p)-\log\phi^*(1)\Big)\left[\frac{n}{p}\right]\\
&+ \sum_{p^2\leqslant n}\Big(\log\phi^*(p^2)-\log\phi^*(p)\Big)\left[\frac{n}{p^2}\right]\\
&+ \sum_{\substack{p^a\leqslant n\\ a\geqslant 3}}\Big(\log\phi^*(p^a)-\log\phi^*(p^{a-1})\Big)\left[\frac{n}{p^a}\right]\\
&:= \Sigma_1(n)+\Sigma_2(n)+\Sigma_3(n).
\end{aligned}$$

By taking logarithm from (1.1), we get

$$\log\phi^*(q)=\log q+\sum_{p\|q}\log\Big(1-\frac{2}{p}\Big)+2\sum_{p^2|q}\log\Big(1-\frac{1}{p}\Big).$$

Also for primes p and integer $a\geqslant 2$ one has

$$\log\phi^*(p)=\log p+\log\Big(1-\frac{2}{p}\Big),\qquad \log\phi^*(p^a)=a\log p+2\log\Big(1-\frac{1}{p}\Big).$$

To approximate $\Sigma_1(n)$ we write

$$\begin{aligned}
\Sigma_1(n) &= \sum_{p\leqslant n}\Big(\log\phi^*(p)-\log\phi^*(1)\Big)\left[\frac{n}{p}\right]\\
&= n\sum_{p\leqslant n}\frac{\log p}{p}+n\sum_{p\leqslant n}\frac{\log(1-\frac{2}{p})}{p}\\
&- \sum_{p\leqslant n}\left\{\frac{n}{p}\right\}\log p-\sum_{p\leqslant n}\left\{\frac{n}{p}\right\}\Big(\log(1-\frac{2}{p})\Big)\\
&:= nS_1(n)+nS_2(n)-S_3(n)-S_4(n).
\end{aligned}$$

$S_1(n)$ is Mertens' first theorem that can be determined in (1.2). To approximate $S_2(n)$ we write

$$S_2(n)=\sum_{p\leqslant n}\frac{\log(1-\frac{2}{p})}{p}=\beta_1+O\Big(\frac{1}{n\log n}\Big).$$

From (1.5) we have

$$S_3(n)=\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}\log p=(1-\gamma)n+O\Big(\frac{n}{\log n}\Big).$$

And,

$$S_4(n)=\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}\Big(\log(1-\frac{2}{p})\Big)=\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}\Big(\frac{-2}{p}+O(\frac{1}{p^2})\Big).$$

To estimate the above series write

$$\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}\frac{1}{p}\ll\sum_{p\leqslant n}\frac{1}{p}\ll\log\log n,\qquad \sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}\frac{1}{p^2}\ll\sum_{p\leqslant n}\frac{1}{p^2}\ll\frac{1}{n\log n}.$$

So, $S_4(n)\ll\log\log n$, thus by combining the estimates of $S_1(n)$, $S_2(n)$, $S_3(n)$ and $S_4(n)$ we get

$$\Sigma_1(n)=n\log n+\Big(E+\beta_1+\gamma-1\Big)n+O\Big(\frac{n}{\log n}\Big).$$

By using similar approach to estimate $\Sigma_1(n)$, series $\Sigma_2(n)$ and $\Sigma_3(n)$ are estimated as follows.

$$\Sigma_2(n)=(\beta_2+2\beta_3-\beta_4)n+O\Big(\sqrt{n}\Big),$$

where

$$\beta_2=\sum_{p^2}\frac{\log p}{p^2},\quad \beta_3=\sum_{p^2}\frac{\log(1-\frac{1}{p})}{p^2},\qquad \beta_4=\sum_{p^2}\frac{\log(1-\frac{2}{p})}{p^2}.$$

And $\Sigma_3(n)=n\beta_5+O\Big(\sqrt{n}\log n\Big)$ where $\beta_5=\sum_{\alpha=2}^{\infty}\sum_{p^\alpha}\frac{\log p}{p^\alpha}$.

Finally, combining approximations $\Sigma_1(n)$, $\Sigma_2(n)$ and $\Sigma_3(n)$ we get

$$\sum_{k\leqslant n}\log\phi^*(k)=n\log n+Kn+O\Big(\frac{n}{\log n}\Big),$$

where

$$K=E+\beta_2+\beta_3+2\beta_4+\beta_5+\beta_6+\gamma-1.$$

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A NEW TYPE OF HADAMARD INEQUALITY AND HARMONICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish a new type of Hadamard inequality for harmonically convex functions. Also we establish several results for convex and harmonically convex functions. In fact, we obtain some results for sum, difference, composition and absolute value of these classes of functions.

1. INTRODUCTION AND PRELIMINARIES

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, [2]. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) by appropriate selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize,

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improve and extend the inequalities (1.1) we refer the reader to the recent papers, see [1,3].

The main purpose of this paper is to introduce the concept of harmonically convex functions (briefly HCF) and establish some results connected with these classes of functions. In the end we obtained some results for sum, difference, composition and absolute value of HCF.

Definition 1.1. Let $I \subset \mathbb{R} - \{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex if

$$f(\frac{xy}{tx+(1-t)y}) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$.

If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Example 1.2. Let $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = cx$, $c > 0$ then f is an HCF.

Proof. Because for all $x, y \in (0,\infty)$ and $t \in [0,1]$, we have $(x-y)^2 \geq 0$ and $t(x-y)^2 - t^2(x-y)^2 \geq 0$, thus

$$c(tx^2 - 2txy + ty^2 - t^2x^2 + 2t^2xy - t^2y^2 + xy) \geq cxy$$

and

$$c(tx + y - ty)(ty + x - tx) \geq cxy.$$

Since $x > 0, y > 0$ and $tx + (1-t)y \neq 0$, $\frac{cxy}{tx+(1-t)y} \leq c(ty + (1-t)x)$. Therefore we have

$$f(\frac{xy}{tx+(1-t)y}) \leq tf(y) + (1-t)f(x).$$

□

In [2], the author gave the definition of HCF and established some Hermite-Hadamard type inequalities for harmonically convex functions as follows.

Theorem 1.3 (Theorem 1 in [2]). *Let $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be an* HCF *and $a, b \in I$ with $a < b$. If f is integrable on $[a,b]$ then the following inequalities hold*

$$f(\frac{2ab}{a+b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

2. Main Results

The main purpose of this section is to establish some results about HCF.

Theorem 2.1. *Let $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be an HCF and $a, b \in I$ with $a < b$. If f is integrable on $[a, b]$ then*

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq \frac{mf(b) + (n-m)f(a)}{n} \tag{2.1}$$

where

$$\lambda_1 = 2m(1 - \frac{m}{n}), \quad \lambda_2 = n - 2m(1 - \frac{m}{n}), \quad m, n \in \mathbb{N}$$

Proof. Since $f : I \longrightarrow \mathbb{R}$ is an HCF, for all $x, y \in I$ (with $0 < t = m/n < 1$ in the inequality (1.2)) we have

$$f(\frac{nxy}{mx + (n-m)y}) \leq \frac{mf(y) + (n-m)f(x)}{n}.$$

Choosing $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$, we get

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{m}{n} f(\frac{ab}{tb + (1-t)a}) + \frac{n-m}{n} f(\frac{ab}{ta + (1-t)b}).$$

Further, by integrating for $t \in [0, 1]$, we have

$$\begin{aligned} f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{1}{n}[m\int_0^1 f(\frac{ab}{tb + (1-t)a})dt \\ + (n-m)\int_0^1 f(\frac{ab}{ta + (1-t)b})dt] \end{aligned} \tag{2.2}$$

Making the change of variables

$$\frac{ab}{tb + (1-t)a} = u, \qquad \frac{-(b-a)ab}{(tb + (1-t)a)^2}dt = du$$

in last integrals in (2.2), we have

$$\int_0^1 f(\frac{ab}{tb + (1-t)a})dt = \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du. \tag{2.3}$$

From (2.2) and (2.3), we have

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \tag{2.4}$$

For the second inequality, use (1.2) with $x = a$,$y = b$ and get

$$f(\frac{ab}{ta+(1-t)b}) \leq \frac{mf(b)+(n-m)f(a)}{n} \tag{2.5}$$

Now, by integrating with respect to t over $[0,1]$ from (2.5), we have

$$\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq \frac{mf(b)+(n-m)f(a)}{n}$$

This completes the proof. $\square$

Proposition 2.2. (i) *Let* $f, g : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ *be* HCF *and* $D_f \cap D_g \neq \varnothing$. *Then* $f+g$,$f-g$ *are* HCF.

(ii) *Let* $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ *be a* HCF, g *be a nondecreasing and convex function such that* $R_f \cap D_g \neq \varnothing$. *Then* $g \circ f$ *is* HCF.

Example 2.3. Let $g(x) = e^x$ and f be an HCF. Then it is clear that $(g \circ f)(x) = e^{f(x)}$ is HCF.

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Keynote Presentations

FUNCTIONAL ANALYSIS; A HISTORICAL PERSPECTIVE

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Abstract. Fourier in his celebrated book, The Analytic Theory of Heat (1822), discussed the first example of what is now known as the problem of (inverse) Fourier transform. About the same time, Niels Abel (1823) offered a solution to the tautochrone problem in the form of an integral equation. More generally, Liouville, in his research on 2nd order linear differential equations (1837) reduced the problem to certain integral equations.

The first rigorous treatment of the general theory of integral equations was given by Ivar Fredholm (1900-1903). Hilbert was attracted to the new theory and published a series of five papers (1904-1906). Along these, the history of Functional Analysis (a name coined by Paul Lévy in 1922) is marked by Lebesgue thesis on integration (1902), Hilbert paper on spectral theory (1906), Fréchet thesis on metric spaces (1906), Riesz papers on classical Banach spaces (1910-1911), Banach thesis on normed spaces (1922), Hahn and Banach papers on duality (1927 and 1929, independent). These were complemented by the pioneering books of Fréchet (1928) and Banach (1932).

We give a glimpse of the development of Functional Analysis by reminding the turning points of each of the above basic steps, as well as the later developments.

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Key words and phrases. history of functional analysis, mathematical analysis, Fourier, Fréchet.

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GEOMETRIC PROPERTIES OF BANACH SPACES AND METRIC FIXED POINT THEORY

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ABSTRACT. This largely expository talk will begin with a brief introduction to historical origins of metric fixed point theory and proceed with a survey of some developments in metric fixed point theory of Banach spaces. I will also discuss some fundamental problems that have been solved, and other questions that remain open.

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